Hölder Continuity of Solutions to the G-Laplace Equation

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Date: 2018-09-22T00:00:00+00:00

Abstract

We establish regularity of solutions to the G-Laplace equation $-\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu$, where μ is a nonnegative Radon measure satisfying $\mu(B_r(x_0)) \le Cr^m$ for any ball $B_r(x_0) \subset\subset \Omega$ with $r \le 1$ and $m > n-1-\delta \ge 0$. The function $g(t)$ is supposed to be nonnegative and C^1-continuous in $[0,+\infty)$, satisfying $g(0) = 0$, and for some positive constants δ and g_0, $\delta \le \frac{tg'(t)}{g(t)} \le g_0, \forall t > 0$, that generalizes the structural conditions of Ladyzhenskaya-Ural' tseva for an elliptic operator.

Full Text

Hölder Continuity of Solutions to the G-Laplace Equation Involving Measures

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$$\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu,$$

where μ is a nonnegative Radon measure satisfying $\mu(B_r(x_0)) \le Cr^m$ for any ball $B_r(x_0) \subset\subset \Omega$ with $r \le 1$ and $m > n-1-\delta \ge 0$. The function $g(t)$ is supposed to be nonnegative and C^1-continuous in $[0,+\infty)$, satisfying $g(0) = 0$, and for some positive constants δ and g_0,

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1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$ $(n \geq 2)$, and μ a nonnegative Radon measure in Ω with $\mu(B_r(x_0)) \leq Cr^m$ for some constant $C > 0$ whenever $B_r(x_0) \subset\subset \Omega$. We consider the equation

$$-\Delta_G u = -\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in } \mathcal{D}'(\Omega),$$

where $G(t) = \int_0^t g(s)ds$, and $g(t)$ is a nonnegative C^1 function in $[0, +\infty)$ satisfying $g(0) = 0$ and the structural condition

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall t > 0,$$

with δ and g_0 being positive constants.

The structural conditions on g were introduced by Lieberman in 1991, representing a natural generalization of the natural conditions of Ladyzhenskaya and Ural' tseva for elliptic equations (see [10]). These conditions imply that the operator Δ_G includes not only the p-Laplace operator Δ_p where $g(t) = t^{p-1}$ and $\delta = g_0 = p - 1$, but also the case of a variable exponent $p = p(t) > 0$:

$$-\Delta_G u = -\mathrm{div}(|\nabla u|^{p(|\nabla u|)-2}\nabla u),$$

corresponding to setting $g(t) = t^{p(t)-1}$, for which the structural condition holds if $\delta \leq t(\ln t)p'(t) + p(t) - 1 \leq g_0$ for all $t > 0$. Another typical example is $g(t) = t^p \log(at + b)$ with $p, a, b > 0$, where in this case $\delta = p$ and $g_0 = p + 1$. Many other examples can be found in [2, 3, 6] and related references.

Under assumption (2), G is an increasing C^2 convex function, which is an N-function satisfying the so-called Δ_2-condition. Thus our class of operators will be considered in the setting of Orlicz spaces. We recall the definitions of Orlicz and Orlicz-Sobolev spaces together with their respective norms (see [1]):

$$L^G(\Omega) = \left\{u \in L^1(\Omega); \int_\Omega G(|u(x)|)dx < +\infty\right\},$$

$$\|u\|_{L^G(\Omega)} = \inf\left\{k > 0; \int_\Omega G\left(\frac{|u(x)|}{k}\right)dx \leq 1\right\},$$

$$W^{1,G}(\Omega) = \{u \in L^G(\Omega); \nabla u \in L^G(\Omega)\},$$

$$\|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}.$$

Under assumption (2), $W^{1,G}(\Omega)$ is a reflexive and separable Banach space (see [1]).

We shall call a solution of (1) any function $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ that satisfies

$$\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla \phi \, dx = \int_\Omega \phi \, d\mu \quad \forall \phi \in \mathcal{D}(\Omega).$$

If $\mu \equiv 0$ in a domain $D \subset \Omega$, we say that u is G-harmonic in D.

We now introduce the regularity background for related elliptic equations involving measures. In 1994, Kilpeläinen considered the p-Laplace operator and proved that if μ satisfies $\mu(B_r) \leq Cr^{n-p+\alpha(p-1)}$ for some positive constants C and $\alpha \in (0,1]$, then any solution of the p-Laplace equation $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu$ is $C^{0,\beta}_{\mathrm{loc}}$-continuous for each $\beta \in (0,\alpha)$ (see [7]). This result was improved by Kilpeläinen and Zhong in 2002, showing that each solution of (3) is in fact Hölder continuous with the same exponent α as in the assumption $\mu(B_r) \leq Cr^{n-p+\alpha(p-1)}$ (see [8]). In 2010, Lyaghfouri extended the p-Laplace problem (3) to the case with variable exponents, i.e., considering $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu$. Under certain assumptions on the function $p(x)$ and the condition $\mu(B_r) \leq Cr^{n-p(x)+\alpha(p(x)-1)}$ for some positive constants C and $\alpha \in (0,1]$, the author proved that any bounded solution of (4) is $C^{0,\alpha}_{\mathrm{loc}}$-continuous with the same exponent α (see [11]).

When focusing on problems governed by the G-Laplacian, Challal and Lyaghfouri proved that if $\mu(B_r(x_0)) \leq Cr^m$ with $m \in [n-1,n)$, then any solution of (1) is $C^{0,\alpha}_{\mathrm{loc}}$-continuous with $\alpha = \frac{m-n+1+\delta}{1+g_0}$. Particularly, if $m = n-1$, then any solution is $C^{0,\alpha}_{\mathrm{loc}}$-continuous for any $\alpha \in (0, \frac{\delta}{1+g_0})$ (see Theorem 3.3 in [3]). In 2011, these regularity results were improved by Challal and Lyaghfouri in [5], showing that any locally bounded solution of (1) is $C^{0,\alpha}_{\mathrm{loc}}$-continuous for any $\alpha \in (0, \frac{m-n+1+\delta}{1+g_0})$ provided that $m > n-1-\delta$. Note that under the assumption of non-decreasing monotonicity on $g(t)$, Zheng, Feng and Zhang obtained local $C^{1,\alpha}$-continuity of solutions for $m > n$ and local Hölder continuity with small exponents for some $m < n$ in 2015 (see [14]).

In this paper, we continue the work of Challal, Lyaghfouri and Zheng et al. by improving the regularity of solutions to equation (1). In particular, we can prove the $C^{0,\alpha}_{\mathrm{loc}}$-continuity of solutions for any $\alpha \in (0,1)$ when $m = n-1$. More precisely, for any $m > n-1-\delta$ and without any monotonicity assumption on $g(t)$, we have the following result.

Theorem 1.1. Assume that μ satisfies (1) with $m > n-1-\delta \geq 0$. Then we have:

(i) If $m > n$, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0, \min\{\sigma, \frac{m-n}{2(1+g_0)}\})$, where σ is the same as in Lemma 2.4.

(ii) If $m \in [n-1,n)$, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$.

(iii) If $n-1-\delta < m < n-1$, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0, \frac{m-n+1+\delta}{1+g_0})$.

Remark 1. In [7], the author proved for the p-Laplacian problem that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$ provided $m = n-1$. In this paper we not only improve the results of [3, 5] and [14], but also extend the problem in [7] to general equations governed by a large class of degenerate and singular elliptic operators.

2. Preliminaries

In this section, we state some auxiliary results that will be used throughout the paper. We begin with some properties of the function G.

Lemma 2.1 ([13, Lemma 2.1, Remark 2.1]). Function G has the following properties: - (G1) G is convex and C^2. - (G2) $\frac{tg(t)}{1+g_0} \leq G(t) \leq tg(t)$. - (G3) $\min\{s^{\delta+1}, s^{g_0+1}\}G(t) \leq G(st) \leq (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t)$ for all $t \geq 0$. - (G4) $G(a+b) \leq 2^{g_0}(1+g_0)(G(a)+G(b))$ for all $a, b > 0$.

For many more properties of G and problems governed by the operator Δ_G, please see [2, 3, 4, 5, 6, 13, 14, 15, 16] and related references.

The following lemmas establish some properties of G-harmonic functions. Throughout this paper, unless otherwise stated, by B_R and B_r we denote balls contained in Ω with the same center, with $B_r \subset\subset B_R \subset\subset \Omega$.

Lemma 2.2 ([13, Theorem 2.3]). Assume $u \in W^{1,G}(\Omega)$. Let h be a weak solution of $\Delta_G h = 0$ in B_R with $h - u \in W_0^{1,G}(B_R)$. Then

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))dx \geq C\left(\int_{A_1} G(|\nabla u - \nabla h|)dx + \int_{A_2} \frac{g(|\nabla u|)}{|\nabla u|}|\nabla u - \nabla h|^2 dx\right),$$

where $A_1 = \{x \in B_R; |\nabla u - \nabla h| \leq 2|\nabla u|\}$, $A_2 = \{x \in B_R; |\nabla u - \nabla h| > 2|\nabla u|\}$ and $C = C(\delta, g_0) > 0$.

Lemma 2.3 ([13, Lemma 2.7]). Let $h \in W^{1,G}(\Omega)$ be a weak solution of $\Delta_G h = 0$. Then $h \in C^{1,\alpha}(\Omega)$. Moreover, there exists $C = C(n, \delta, g_0) > 0$ such that for every ball $B_r \subset\subset \Omega$ and every $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|h\|_{L^\infty(B_{2r}(x_0))}) > 0$ such that

$$\int_{B_r} G(|\nabla h|)dx \leq Cr^\lambda.$$

Let $(u)_r = \frac{1}{|B_r|}\int_{B_r} u\,dx$ be the average value of u on the ball B_r. We have:

Lemma 2.4 (Comparison with G-harmonic functions [14, Lemma 3.1]). Assume $u \in W^{1,G}(B_R)$. Let $h \in W^{1,G}(B_R)$ be a weak solution of $\Delta_G h = 0$ in B_R. Then there exists $\sigma \in (0,1)$ and $C = C(n, \delta, g_0) > 0$ such that for each $0 < r \leq R$, there holds

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)dx \leq C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)dx + C\int_{B_R} G(|\nabla u - \nabla h|)dx.$$

Lemma 2.5 ([9, Lemma 2.7]). Let $\phi(s)$ be a non-negative and non-decreasing function. Suppose that

$$\phi(r) \leq C_1\left(\frac{r}{R}\right)^\alpha \phi(R) + C_1 R^\beta,$$

for all $r \leq R \leq R_0$, with α, β and C_1 positive constants. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $C_2 = C_2(C_1, \alpha, \beta, \tau)$ such that for all $r \leq R \leq R_0$ we have $\phi(r) \leq C_2 r^\tau$.

3. Proof of Theorem 1.1

Lemma 3.1. Assume $u \in W^{1,G}(\Omega)$. Let $B_R \subset\subset \Omega$ and $h \in W^{1,G}(B_R)$ be a weak solution of $\Delta_G h = 0$ in B_R with $h - u \in W_0^{1,G}(B_R)$. Then for any $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|u\|_{L^\infty(B_{2R/3})}) > 0$ such that

$$\int_{B_R} G(|\nabla u - \nabla h|)dx \leq CR^m + CR^\lambda,$$

where λ is the same as in Lemma 2.3.

Proof. First, convexity of G gives

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))dx \leq \int_{B_R} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \cdot (\nabla u - \nabla h)dx = \int_{B_R} (u - h)d\mu \leq C\mu(B_R) \leq CR^m,$$

where we used the boundedness of u which forces h to be bounded as well.

Let A_1 and A_2 be defined as in Lemma 2.2. By Lemma 2.2, there exists a constant $C = C(\delta, g_0) > 0$ such that

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))dx \geq C \left(\int_{A_1} G(|\nabla u - \nabla h|)dx + \int_{A_2} \frac{g(|\nabla u|)}{|\nabla u|} |\nabla u - \nabla h|^2 dx \right).$$

By (G2), $G(t)$ is increasing in $t > 0$. It follows from (G2), (G3), (6), (8) and Lemma 2.2 that

$$\int_{B_R} G(|\nabla u - \nabla h|)dx = \int_{A_1} G(|\nabla u - \nabla h|)dx + \int_{A_2} G(|\nabla u - \nabla h|)dx$$

$$\leq C \int_{B_R} (G(|\nabla u|) - G(|\nabla h|))dx + CR^m + CR^\lambda \leq CR^m + CR^\lambda,$$

where in the last inequality we used $(a+b)^\gamma \leq a^\gamma + b^\gamma$ for any $a \geq 0, b \geq 0$ and $\gamma \in (0, 1)$. By (7) and (9), we have

$$\int_{B_R} G(|\nabla u - \nabla h|)dx = \int_{A_1} G(|\nabla u - \nabla h|)dx + \int_{A_2} G(|\nabla u - \nabla h|)dx \leq CR^m + CR^\lambda.$$

Proof of Theorem 1.1. Let h be a G-harmonic function in B_R that agrees with u on the boundary, i.e., $\Delta_G h = 0$ in B_R and $h - u \in W_0^{1,G}(B_R)$. By Lemma 2.4 and Lemma 3.1, for any $r \leq R$ there holds

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)dx \leq C \left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)dx + C \int_{B_R} G(|\nabla u - \nabla h|)dx$$

$$\leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u-(\nabla u)_R|)dx + CR^m + CR^\lambda,$$

where λ is an arbitrary constant in $(0,n)$.

(i) If $m>n$, then we have

$$\int_{B_r} G(|\nabla u-(\nabla u)_r|)dx \leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u-(\nabla u)_R|)dx + CR^{\frac{m+\lambda}{2}}.$$

Since $m>n$ and λ is an arbitrary constant in $(0,n)$, one may choose λ satisfying $\frac{m+\lambda}{2}>n$. In view of Lemma 2.5, we conclude that for any $\tau<\min\{\sigma,\frac{m+\lambda}{2}-n\}$ there holds

$$\int_{B_r} G(|\nabla u-(\nabla u)_r|)dx \leq Cr^{n+\tau}, \quad \forall r\leq R.$$

Now we claim that

$$\int_{B_r} |\nabla u-(\nabla u)_r|dx \leq Cr^{n+\frac{\tau}{1+g_0}}, \quad \forall r\leq R.$$

Indeed, for r satisfying $r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|dx \leq r^{\frac{\tau}{1+g_0}}$, the claim holds with $C=1$. For r satisfying $r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|dx > r^{\frac{\tau}{1+g_0}}$, we infer from the increasing monotonicity of $G(t)$ in $t>0$ that

$$G\left(r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|dx\right) \geq G\left(r^{\frac{\tau}{1+g_0}}\right).$$

It follows from (G2) and (G3) that

$$\int_{B_r} |\nabla u-(\nabla u)_r|dx \leq Cr^{n+\frac{\tau}{1+g_0}}.$$

Note that convexity of G and (10) implies that

$$G\left(\frac{1}{|B_r|}\int_{B_r}|\nabla u-(\nabla u)_r|dx\right) \leq \frac{1}{|B_r|}\int_{B_r} G(|\nabla u-(\nabla u)_r|)dx \leq Cr^\tau.$$

By (G3), (12) and (13), one may get

$$\int_{B_r} |\nabla u-(\nabla u)_r|dx \leq Cr^{n+\frac{\tau}{1+g_0}},$$

where C depends only on $g(1)$, g_0 and the volume of the unit ball. Now we have proven that (11) holds for any $r\leq R$. Thus $u\in C^{1;\alpha}_{\text{loc}}(\Omega)$ by Campanato' s embedding theorem. Due to the arbitrariness of $\lambda\in(0,n)$, we can conclude (i) of Theorem 1.1 by letting $\lambda\to n$.

(ii) If $m \in [n-1, n]$, we only prove for $m = n-1$ due to the fact that $\mu(B_r) \le Cr^m \le Cr^{n-1}$ for small r. By (G4), Lemma 2.3 and Lemma 3.1, we infer

$$\int_{B_r} G(|\nabla u|)dx \le C\int_{B_r} G(|\nabla u - \nabla h|)dx + C\int_{B_r} G(|\nabla h|)dx \le Cr^m + Cr^\lambda \le Cr^m,$$

where in the last inequality we let $n > \lambda > n-1 = m$.

We claim that for any $r \le R < 1$ with $B_R \subset\subset \Omega$ and some positive constant C independent of r, there holds

$$\int_{B_r} |\nabla u|dx \le Cr^{n-1+\alpha_0},$$

with some $\alpha_0 \in (0,1)$.

Indeed, for $r \le R$ satisfying $r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|dx \le 1$, the estimate holds with $C = 1$. For $r \le R$ satisfying $r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|dx \ge 1$, due to the increasing monotonicity of $F(t) = G(t) - G(1)t$ for $t \ge 1$, it follows that

$$G\left(r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|dx\right) \ge G(1)\cdot r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|dx.$$

Then we have

$$\int_{B_r}|\nabla u|dx \le Cr^{n-1+\alpha_0}\left(r^{1-\alpha_0}\right)^{1+\delta} G\left(r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|dx\right) \le Cr^{n-1+\alpha_0+(1-\alpha_0)(1+\delta)}\cdot r^{-n}\cdot r^m = Cr^{n-1+\alpha_0+(1-\alpha$$

Combining (15) and (16), we may choose α_0 satisfying $\alpha_0 = \alpha_0 + (1-\alpha_0)(1+\delta) + m - n$, i.e., $\alpha_0 = 1 - \frac{n-m}{1+\delta}$. For $m = n-1$, we conclude that $u \in C^{0,\alpha_0}_{\text{loc}}(\Omega)$ by Morrey' s theorem (see page 30, [12]) with $\alpha_0 = \frac{\delta}{1+\delta}$ such that the estimate holds for all $r \le R$.

Note that $\inf_{B_r} u \le \inf_{B_r} h$ (see the proof of Theorem 3.3 in [3]). Then by (5) and Lemma 2.3, we have for λ larger than $m + \alpha_0$:

$$\int_{B_r} G(|\nabla u|)dx \le \int_{B_r}(u-h)d\mu + \int_{B_r} G(|\nabla h|)dx \le (\sup_{B_r} u - \inf_{B_r} h)\mu(B_r) + Cr^\lambda \le \operatorname{osc}(u, B_r)r^m + Cr^\lambda \le Cr^{\alpha_0+m} + C$$

where $\operatorname{osc}(u, B_r) = \sup_{B_r} u - \inf_{B_r} u$. Arguing as in (14), we get $u \in C^{0,\alpha_1}_{\text{loc}}(\Omega)$ with $\alpha_1 = 1 - \frac{n-(m+\alpha_0)}{1+\delta}$. Repeating this process, we obtain $u \in C^{0,\alpha_k}_{\text{loc}}(\Omega)$ with

$$\alpha_k = \frac{\alpha_0}{(1+\delta)^k} + \frac{1+\delta+m-n}{1+\delta}\sum_{j=0}^{k-1}\frac{1}{(1+\delta)^j},$$

which leads to $\lim_{k\to\infty}\alpha_k = 1$, and the result follows.

(iii) If $n-1-\delta < m < n-1$, checking the proof and repeating the process as above, we may get $\alpha_0 = 1 - \frac{n-m}{1+\delta}$, $\alpha_1 = \frac{1+\delta+m-n}{1+\delta} + \frac{\alpha_0}{1+\delta}$, ···, $\alpha_k = \frac{1+\delta+m-n}{1+\delta} + \frac{\alpha_{k-1}}{1+\delta}$. Finally, one has $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0, \frac{1+\delta+m-n}{1+\delta})$.

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Received xxxx 20xx; revised xxxx 20xx.

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Note: Figure translations are in progress. See original paper for figures.