The Obstacle Problem for Non-Coercive Equations with Lower-Order Terms and L^1- Data

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Full Text

Preamble

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The aim of this paper is to study the obstacle problem associated with an elliptic operator having degenerate coercivity, and with a low order term and L^1-data. We prove the existence of an entropy solution to the obstacle problem and show its continuous dependence on the L^1-data in $W^{1,q}(\Omega)$ with some $q > 1$.

Key words: obstacle problem; non-coercive equation; entropy solution; L^1-data; lower order term.

1.1 Problem setting and main result

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $1 < p < +\infty$ and $\theta \geq 0$. Given functions $g, \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and data $f \in L^1(\Omega)$, the aim of this paper is to study the obstacle problem for nonlinear non-coercive elliptic equations with lower order term, governed by the operator

$Au = -\text{div }a(x, \nabla u)(1 + |u|)^{\theta(p-1)} + b|u|^{r-2}u,$

where $b > 0$ is a constant, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, satisfying the following conditions:

$a(x, \xi) \cdot \xi \geq \alpha|\xi|^p,$
$|a(x, \xi)| \leq \beta(j(x) + |\xi|^{p-1}),$
$(a(x, \xi) - a(x, \eta))(\xi - \eta) > 0,$
$|a(x, \xi) - a(x, \zeta)| \leq \gamma\{|\xi - \zeta|^{p-1}, (1 + |\xi| + |\zeta|)^{p-2}|\xi - \zeta|\}$, if $p \geq 2$

for almost every x in Ω, and for every ξ, η, ζ in $\mathbb{R}^N$ with $\xi \neq \eta$, where $\alpha, \beta, \gamma > 0$ are constants, and j is a nonnegative function in $L^{p'}(\Omega)$. If f has fine regularity, i.e., $f \in W^{-1,p'}(\Omega)$, the obstacle problem corresponding to (f, ψ, g) can be formulated in terms of

$\int_\Omega a(x, \nabla u)(1+|u|)^{\theta(p-1)} \cdot \nabla(u-v)dx + \int_\Omega b|u|^{r-2}u(u-v)dx \leq \int_\Omega f(u-v)dx, \forall v \in K_{g,\psi} \cap L^\infty(\Omega),$

whenever $1 \leq r < p$ and the convex subset

$K_{g,\psi} = \{v \in W^{1,p}(\Omega); v - g \in W_0^{1,p}(\Omega), v \geq \psi, \text{ a.e. in } \Omega\}$

is nonempty. However, if $f \in L^1(\Omega)$, (6) is not well-defined, and following [1,3,6] etc., we are led to the more general definition of a solution to the obstacle problem, using the truncation function $T_s(t) = \max\{-s, \min\{s, t\}\}$, $s, t \in \mathbb{R}$.

Definition 1 An entropy solution of the obstacle problem associated corresponding to operator A and functions (f, ψ, g) with $f \in L^1(\Omega)$ is a measurable function u such that $u \geq \psi$ a.e. in Ω, $(1+|u|)^{\theta(p-1)} \in (L^1(\Omega))^N$, $|u|^{r-1} \in L^1(\Omega)$, and, for every $s > 0$, $T_s(u) - T_s(g) \in W_0^{1,p}(\Omega)$ and

$\int_\Omega a(x, \nabla u)(1+|u|)^{\theta(p-1)} \cdot \nabla(T_s(u-v))dx + \int_\Omega b|u|^{r-2}uT_s(u-v)dx \leq \int_\Omega fT_s(u-v)dx, \forall v \in K_{g,\psi} \cap L^\infty(\Omega).$

Observe that no global integrability condition is required on u nor on its gradient in the definition. As pointed out in [3,9], if $T_s(u) \in W^{1,p}(\Omega)$ for all $s > 0$, then there exists a unique measurable vector field $U : \Omega \to \mathbb{R}^N$ such that $\nabla(T_s(u)) = \chi_{|u|<s}U$ a.e. in Ω, $s > 0$, which, in fact, coincides with the standard distributional gradient of u whenever $u \in W^{1,1}(\Omega)$.

Before stating the main result, we make some basic assumptions throughout this paper, i.e., without special statements, we always assume that

$1 < p < N, 1 \leq r < p, 0 \leq \theta < \min\{N/(N-1)-1/(p-1), (p-r)/(p-1)\}, b > 0,$

and $\psi, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $(\psi - g)^+ \in W_0^{1,p}(\Omega)$ such that $K_{g,\psi} \neq \emptyset$. The following theorem is the main result obtained in this paper.

Theorem 1 Let $f \in L^1(\Omega)$. Then there exists at least one entropy solution u of the obstacle problem associated with (f, ψ, g). In addition, u depends continuously on f, i.e., if $f_n \to f$ in $L^1(\Omega)$ and u_n is a solution to the obstacle problem associated with (f_n, ψ, g), then $u_n \to u$ in $W^{1,q}(\Omega)$, $\forall q \in (1, N(p -$

$1)(1-\theta)/(N-1-\theta(p-1)))$ if $1 \leq r < \min\{2N-1/N-1, p\}$, and $\forall q \in (N(r-1)/(N+r-1), N(p-1)(1-\theta)/(N-1-\theta(p-1)))$ if $2N-1/N-1 \leq r < p$.

Some comments and remarks

Consider the Dirichlet boundary value problem of the form

$-\text{div}\ (|\nabla u|^{p-2}\nabla u/(1+|u|)^{\theta(p-1)}) + bu = f$ in Ω,
$u = 0$ on $\partial\Omega$,

with $p > 1, \theta \in (0,1], b \geq 0, f \in L^1(\Omega)$. The term $|\nabla u|^{p-2}\nabla u/(1+|u|)^{\theta(p-1)}$ is not coercive if u is very large. Due to this, the classical methods used to prove the existence of a solution for elliptic equations, i.e., [15], cannot be applied even if $b = 0$ and the data f is regular. Moreover, $|\nabla u|^{p-2}\nabla u/(1+|u|)^{\theta(p-1)}$ and u and f are only in $L^1(\Omega)$, not in $W^{-1,p'}(\Omega)$. All these characteristics prevent us from employing the duality argument [18] or nonlinear monotone operator theory [19] directly. To overcome these difficulties, by cutting the noncoercivity term and using the technique of approximation, a pseudomonotone and coercive differential operator on $W^{1,p}(\Omega)$ can be applied to establish a priori estimates on approximating solutions. As a result, existence of solutions, or entropy solutions, can be obtained by taking limits for $f \in L^m(\Omega), m \geq 1$ and $b > 0$ due to the almost everywhere convergence for the gradients of the approximating solutions, see e.g., [4,7,10–12,16] (see also [1,2,8,13,14,17] for $b = 0$).

Motivated by the study on the non-coercive elliptic equations (9), we consider in this paper the obstacle problem governed by (1) and functions (f, ψ, g) with $f \in L^1(\Omega)$. We prove the existence of an entropy solution and show its continuous dependence on the L^1-data in $W^{1,q}(\Omega)$ with some $q \in (1, p)$.

In the following, we give some remarks on our main result and inequalities that will be needed in the proofs. Some notations are provided at the end of this subsection.

Remark 1 (i) $0 \leq \theta < \min\{N/(N-1) - 1/(p-1), (p-r)/(p-1)\} \Rightarrow r-1 < (1-\theta)(p-1) < N(p-1)(1-\theta)/(N-1-\theta(p-1))$. Therefore Theorem 1 guarantees $|u|^{r-1} \in L^1(\Omega)$, and the second integration in (7) makes sense.

(ii) We will show that $a(x, \nabla u)/(1+|u|)^{\theta(p-1)} \in (L^1(\Omega))^N$ in Proposition 4. Therefore, the first integration in (7) makes sense.

(iii) Note that $\theta < (p-r)/(p-1) \Leftrightarrow N(p-1)(1-\theta)/(N-1-\theta(p-1)) > N(r-1)/(N+r-1)$, while $2N-1/N-1 \leq r$ gives $N(r-1)/(N+r-1) \geq 1$. Thus $u_n \to u$ in $W^{1,q}(\Omega)$ for all $q \in (1, N(p-1)(1-\theta)/(N-1-\theta(p-1)))$ if $2N-1/N-1 \leq r < p$. Indeed, $\theta < (p-r)/(p-1) < N/(N-1)-1/(p-1) \Leftrightarrow N(p-1)(1-\theta)/(N-1-\theta(p-1)) > N(r-1)/(N+r-1)$.

(iv) $r-1 < Nq/(N-q)$ for any $q \in (1, N(p-1)(1-\theta)/(N-1-\theta(p-1)))$. Indeed, by $1 \leq r < 2N-1/N-1$, there holds $r-1 < N/(N-1) < Nq/(N-q)$. For $r \geq 2N-1/N-1$, it suffices to note that $q > N(r-1)/(N+r-1) \Leftrightarrow r-1 < Nq/(N-q)$.

(v) $q < p$. Indeed, $0 \leq \theta < N/(N-1) - 1/(p-1) < (N-1)/(p-1) \Rightarrow N(p-1)(1-\theta)/(N-1-\theta(p-1)) < p$.

Remark 2 Checking proofs of this paper, one may find that, for $b = 0$, (8) holds with $u_n \to u$ in $W^{1,q}(\Omega)$, $\forall q \in (1, N(p-1)(1-\theta)/(N-1-\theta(p-1)))$. Indeed, it suffices to set $r = 1$ in the proofs.

Notations $\|u\|_p = \|u\|_{L^p(\Omega)}$ is the norm of $L^p(\Omega)$, where $1 \leq p \leq \infty$. $\|u\|_{1,p} = \|u\|_{W^{1,p}(\Omega)}$ is the norm of $W^{1,p}(\Omega)$, where $1 < p < \infty$. $p' = p/(p-1)$ with $1 < p < \infty$. $u^+ = \max\{u, 0\}, u^- = (-u)^+$, if u is a real-valued function. C is a constant, which may be different from each other. $\{u > s\} = \{x \in \Omega; u(x) > s\}$. $\{u \leq s\} = \Omega \quad \{u > s\}$. $\{u < s\} = \{x \in \Omega; u(x) < s\}$. $\{u \geq s\} = \Omega \quad \{u < s\}$. $\{u = s\} = \{x \in \Omega; u(x) = s\}$. $\{t \leq u < s\} = \{u \geq t\} \cap \{u < s\}$. L^N is the Lebesgue measure of $\mathbb{R}^N$. $|E| = L^N(E)$ for a measurable set E.

2 Lemmas on entropy solutions

It is worthy to note that, for any smooth function f_n, there exists at least one solution to the obstacle problem (6). Indeed, one can proceed exactly as in [1,12] to obtain $W^{1,p}$-solutions due to the assumptions (2)-(4) on a and $r - 1 < p$. These solutions, in particular, are also entropy solutions. In this section we establish several auxiliary results on convergence of sequences of entropy solutions when $f_n \to f$ in $L^1(\Omega)$.

Lemma 2 Let $v_0 \in K_{g,\psi} \cap L^\infty(\Omega)$, and let u be an entropy solution of the obstacle problem associated with (f, ψ, g). Then, we have

$\int_{|u|<t} |\nabla u|^p/(1+|u|)^{\theta(p-1)} dx \leq C(1+t^r), \forall t > 0,$

where C is a positive constant depending only on $\alpha, \beta, p, r, b, \|j\|_{p'}, \|\nabla v_0\|_p, \|v_0\|_\infty$, and $\|f\|_1$.

Proof. Take v_0 as a test function in (7). For t large enough such that $t - \|v_0\|_\infty > 0$, we get

$\int_{|v_0-u|<t} a(x, \nabla u) \cdot \nabla u/(1+|u|)^{\theta(p-1)} dx \leq \int_{|v_0-u|<t} a(x, \nabla u) \cdot \nabla v_0/(1+|u|)^{\theta(p-1)} dx + \int_\Omega (f - b|u|^{r-2}u) T_t(u - v_0) dx.$

We estimate each integral on the right-hand side of (11). It follows from (3) and Young's inequality with $\varepsilon > 0$ that

$\int_{|v_0-u|<t} a(x, \nabla u) \cdot \nabla v_0/(1+|u|)^{\theta(p-1)} dx \leq \int_{|v_0-u|<t} \beta(|j| + |\nabla u|^{p-1})|\nabla v_0|/(1+|u|)^{\theta(p-1)} dx \leq \int_{|v_0-u|<t} \beta\varepsilon(|j|^{p'} + |\nabla u|^p)/(1+|u|)^{\theta(p-1)} dx + C(\|j\|_{p'}^{p'} + \|\nabla v_0\|_p^p).$

Note that on the set $\{|u - v_0| \leq t\}$, we have $||u|^{r-2} u T_t(u - v_0)| \leq t|t + \|v_0\|_\infty|^{r-1} \leq C(1+t^r)$, where C is a constant depending only on $r, \|v_0\|_\infty$.

On the set $\{|u - v_0| > t\}$, we have $|u| \geq t - \|v_0\|_\infty > 0$, thus u and $T_t(u - v_0)$ have the same sign. It follows that

$\int_{|u-v_0|>t} b|u|^{r-2}uT_t(u-v_0)dx \leq 0.$

Combining these estimates, we obtain

$\int_{|v_0-u|<t} |\nabla u|^p/(1+|u|)^{\theta(p-1)}dx \leq C(1+t^r).$

Replacing t with $t+\|v_0\|_\infty$ in (17) and noting that $\{|u|<t\} \subset \{|v_0-u| < t+\|v_0\|_\infty\}$, one may obtain the desired result.

In the rest of this section, let $\{u_n\}$ be a sequence of entropy solutions of the obstacle problem associated with (f_n, ψ, g) and assume that $f_n \to f$ in $L^1(\Omega)$ and $\|f_n\|_1 \leq \|f\|_1 + 1$.

Lemma 3 There exists a measurable function u such that $u_n \to u$ in measure, and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W^{1,p}(\Omega)$ for any $k>0$. Thus $T_k(u_n) \to T_k(u)$ strongly in $L^p(\Omega)$ and a.e. in Ω.

Proof. Let s, t and ε be positive numbers. One may verify that for every $m, n \geq 1$,

$L^N(\{|u_n-u_m|>s\}) \leq L^N(\{|u_n|>t\}) + L^N(\{|u_m|>t\}) + L^N(\{|T_k(u_n) - T_k(u_m)|>s\}),$

$L^N(\{|u_n|>t\}) = \int_{\{|u_n|>t\}} 1dx \leq (1/t^p)\int_\Omega |T_t(u_n)|^p dx.$

Due to $v_0 = g + (\psi-g)^+ \in K_{g,\psi} \cap L^\infty(\Omega)$, by Lemma 2, one has

$\int_\Omega |\nabla T_t(u_n)|^p dx = \int_{\{|u_n|<t\}} |\nabla u_n|^p dx \leq C(1+t)^{\theta(p-1)}(1+t^r).$

Note that $T_t(u_n) - T_t(g) \in W_0^{1,p}(\Omega)$. By (19), (20) and Poincaré's inequality, for every $t > \|g\|_\infty$ and for some positive constant C independent of n and t, there holds

$L^N(\{|u_n|>t\}) \leq (1/t^p)\int_\Omega |T_t(u_n)|^p dx \leq (1/t^p)\int_\Omega |T_t(u_n) - T_t(g)|^p dx + (1/t^p)\int_\Omega |T_t(g)|^p dx \leq (C/t^p)\int_\Omega |\nabla T_t(u_n) - \nabla T_t(g)|^p dx + (1/t^p)\|g\|_p^p \leq (C/t^p)\int_\Omega |\nabla T_t(u_n)|^p dx + (1/t^p)\|g\|_p^p \leq C(1+t^{r+\theta(p-1)})/t^p + (1/t^p)\|g\|_p^p.$

Since $0 \leq \theta < (p-r)/(p-1)$, there exists $t_\varepsilon > 0$ such that

$L^N(\{|u_n|>t\}) < \varepsilon/3, \forall t \geq t_\varepsilon, \forall n \geq 1.$

Now we have, as in (19),

$L^N(\{|T_{t_\varepsilon}(u_n) - T_{t_\varepsilon}(u_m)|>s\}) = \int_{\{|T_{t_\varepsilon}(u_n)-T_{t_\varepsilon}(u_m)|>s\}} 1dx \leq (1/s^p)\int_\Omega |T_{t_\varepsilon}(u_n) - T_{t_\varepsilon}(u_m)|^p dx.$

Using (20) and the fact that $T_t(u_n) - T_t(g) \in W_0^{1,p}(\Omega)$ again, we see that $\{T_{t_\varepsilon}(u_n)\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Thus, up to a subsequence, $\{T_{t_\varepsilon}(u_n)\}$ converges strongly in $L^p(\Omega)$. Taking into account (22), there exists $n_0 = n_0(t_\varepsilon, s) \geq 1$ such that

$L^N(\{|T_{t_\varepsilon}(u_n) - T_{t_\varepsilon}(u_m)|>s\}) < \varepsilon/3, \forall n, m \geq n_0.$

Combining (18), (21) and (23), we obtain

$L^N(\{|u_n - u_m| > s\}) < \varepsilon, \forall n, m \geq n_0$.

Hence $\{u_n\}$ is a Cauchy sequence in measure, and therefore there exists a measurable function u such that $u_n \to u$ in measure.

The remainder of the lemma is a consequence of the fact that $\{T_k(u_n)\}$ is a bounded sequence in $W^{1,p}(\Omega)$.

Proposition 4 There exists a subsequence of $\{u_n\}$ and a measurable function u such that for each q given in (8), we have $u_n \to u$ strongly in $W^{1,q}(\Omega)$. If moreover $0 \leq \theta < \min\{(N-p+1)/(N-1), N/(N-1)-1/(p-1), (p-r)/(p-1)\}$, then

$a(x, \nabla u_n)/(1 + |u_n|)^{\theta(p-1)} \to a(x, \nabla u)/(1 + |u|)^{\theta(p-1)}$ strongly in $(L^1(\Omega))^N$.

To prove Proposition 4, we need two preliminary lemmas.

Lemma 5 There exists a subsequence of $\{u_n\}$ and a measurable function u such that for each q given in (8), we have $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$, and $u_n \to u$ strongly in $L^q(\Omega)$.

Proof. Let $k > 0$ and $n \geq 1$. Define $D_k = \{|u_n| \leq k\}$ and $B_k = \{k \leq |u_n| < k+1\}$. Using Lemma 2 with $v_0 = g + (\psi - g)^+$, we get

$\int_{D_k} |\nabla u_n|^p/(1 + |u_n|)^{\theta(p-1)} dx \leq C(1 + k^r)$,

where C is a positive constant depending only on $\alpha, \beta, b, p, r, \|j\|_{p'}, \|f\|_1, \|\nabla v_0\|_p$, and $\|v_0\|_\infty$.

Using the function $T_k(u_n)$ for $k > \{\|g\|_\infty, \|\psi\|_\infty\}$ as a test function for the problem associated with (f_n, ψ, g), we obtain

$\int_\Omega a(x, \nabla u_n) \cdot \nabla(T_1(u_n - T_k(u_n)))/(1 + |u_n|)^{\theta(p-1)} dx + \int_\Omega b|u_n|^{r-2} u_n T_1(u_n - T_k(u_n)) dx \leq \int_\Omega f_n T_1(u_n - T_k(u_n)) dx$,

which together with (2) gives

$\int_{B_k} \alpha|\nabla u_n|^p/(1 + |u_n|)^{\theta(p-1)} dx + \int_{\{|u_n| \geq k+1\}} b|u_n|^{r-2} u_n T_1(u_n - T_k(u_n)) dx \leq \|f_n\|_1 \leq \|f\|_1 + 1$.

Note that on the set $\{|u_n| \geq k+1\}$, u_n and $T_1(u_n - T_k(u_n))$ have the same sign. Then

$\int_{\{|u_n| \geq k+1\}} b|u_n|^{r-2} u_n T_1(u_n - T_k(u_n)) dx = \int_{\{|u_n| \geq k+1\}} b|u_n|^{r-1} dx \geq 0$.

Thus we have

$\int_{B_k} |\nabla u_n|^p/(1 + |u_n|)^{\theta(p-1)} dx \leq C(1 + k^r)$.

Let $s = q\theta(p-1)/(p-q)$. Note that $q < p$ and $ps/(p-q) < q^*$. For any $k > 0$, we estimate $\int_\Omega |\nabla u_n|^q dx$ as follows:

$\int_\Omega |\nabla u_n|^q dx = \int_\Omega |\nabla u_n|^q (1+|u_n|)^s \cdot (1+|u_n|)^{-s} dx \leq (\int_\Omega |\nabla u_n|^p/(1+|u_n|)^{\theta(p-1)} dx)^{q/p} (\int_\Omega (1+|u_n|)^{ps/(p-q)} dx)^{(p-q)/p} \leq C(1+k^r)^{q/p} (\int_\Omega |u_n|^{q^*} dx + |\Omega|)^{(p-q)/p}$.

Summing up from $k = k_0$ to K and using Hölder's inequality, one obtains uniform bounds. Letting $K \to \infty$, we deduce that $\|\nabla u_n\|_q$ and $\|u_n\|_{q^*}$ are uniformly bounded independently of n. Particularly, u_n is bounded in $W^{1,q}(\Omega)$. Therefore, there exists a subsequence of $\{u_n\}$ and a function $v \in W^{1,q}(\Omega)$ such that $u_n \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$, $u_n \to v$ strongly in $L^q(\Omega)$ and a.e. in Ω. By Lemma 3, $u_n \to u$ in measure in Ω, we conclude that $u = v$ and $u \in W^{1,q}(\Omega)$.

Lemma 6 There exists a subsequence of $\{u_n\}$ and a measurable function u such that ∇u_n converges almost everywhere in Ω to ∇u.

Proof. The proof is quite similar to Theorem 4.1 in [1], which can also be found in [5]. Here we sketch only the main steps due to slight modifications. For $r_2 > 1$, let $\lambda = q/p < 1$, where q is the same as in Lemma 5. Define

$A(x,u,\xi) = a(x,\xi)/(1+|u|)^{\theta(p-1)}$ (for the sake of simplicity, we omit the dependence of $A(x,u,\xi)$ on x)

and

$I(n) = \int_\Omega ((A(u_n,\nabla u_n) - A(u_n,\nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$

We fix $k > 0$ and split the integral in $I(n)$ on the sets $\{|u| > k\}$ and $\{|u| \leq k\}$, obtaining

$I_1(n,k) = \int_{\{|u|>k\}} ((A(u_n,\nabla u_n) - A(u_n,\nabla u)) \cdot \nabla(u_n - u))^\lambda dx,$
$I_2(n,k) = \int_{\{|u|\leq k\}} ((A(u_n,\nabla u_n) - A(u_n,\nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$

For $I_2(n,k)$, one has

$I_2(n,k) \leq I_3(n,k) = \int_\Omega ((A_n(u_n,\nabla u_n) - A_n(u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx.$

Fix $h > 0$ and split $I_3(n,k)$ on the sets $\{|u_n - T_k(u)| > h\}$ and $\{|u_n - T_k(u)| \leq h\}$, obtaining

$I_4(n,k,h) = \int_{\{|u_n - T_k(u)|>h\}} ((A_n(u_n,\nabla u_n) - A_n(u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx,$
$I_5(n,k,h) = \int_{\{|u_n - T_k(u)|\leq h\}} ((A_n(u_n,\nabla u_n) - A_n(u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx$
$\leq |\Omega|^{1-\lambda} (\int_\Omega (A_n(u_n,\nabla u_n) - A_n(u_n,\nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)) dx)^\lambda$
$= |\Omega|^{1-\lambda} (I_6(n,k,h))^\lambda.$

For $I_6(n,k,h)$, it can be split as the difference $I_7(n,k,h) - I_8(n,k,h)$ where

$I_7(n,k,h) = \int_\Omega A(u_n,\nabla u_n) \cdot \nabla T_h(u_n - T_k(u)) dx,$
$I_8(n,k,h) = \int_\Omega A(u_n,\nabla T_k(u)) \cdot \nabla T_h(u_n - T_k(u)) dx.$

Note that $|\nabla u_n|$ is bounded in $L^q(\Omega)$ and $\lambda p r_2 = q$. Thanks to Lemma 3 and Lemma 5, one may get in the same way as Theorem 4.1 in [1] that

$\lim_{k\to\infty} \limsup_{n\to\infty} I_1(n,k) = 0$,
$\lim_{h\to 0} \limsup_{n\to\infty} I_4(n,k,h) = 0$,
$\lim_{n\to\infty} I_8(n,k,h) = 0$.

For $I_7(n,k,h)$, let $k > \max\{\|g\|_\infty, \|\psi\|_\infty\}$ and $n \geq h+k$. Take $T_k(u)$ as a test function for (7), obtaining

$I_7(n,k,h) \leq \int_\Omega f_n T_h(u_n - T_k(u))dx + \int_\Omega b|u_n|^{r-2}u_n T_h(u_n - T_k(u))dx$.

Note that $r-1 < q^*$, and $\int_\Omega |u_n|^{q^*}dx$ is uniformly bounded (see the proof of Lemma 5), thus $|u_n|^{r-2}u_n$ converges strongly in $L^1(\Omega)$. Therefore we have

$\lim_{n\to\infty} \int_\Omega |u_n|^{r-2}u_n T_h(u_n - T_k(u))dx = \int_\Omega |u|^{r-2}u T_h(u - T_k(u))dx$.

Then using the strong convergence of f_n in $L^1(\Omega)$, one has

$\lim_{n\to\infty} I_7(n,k,h) \leq \int_\Omega f T_h(u - T_k(u))dx + \int_\Omega b|u|^{r-2}u T_h(u - T_k(u))dx$.

It follows that $I_7(n,k,h) \leq 0$. Putting together all the limits and noting that $I(n) \geq 0$, we have $I(n) = 0$.

The same arguments as [1] give that, up to subsequence, $\nabla u_n(x) \to \nabla u(x)$ a.e.

Proof of Proposition 4 We shall prove that ∇u_n converges strongly to ∇u in $L^q(\Omega)$ for each q given by (8). To do that, we will apply Vitali's Theorem, using the fact that by Lemma 5, ∇u_n is bounded in $L^q(\Omega)$ for each q given by (8). So let $s \in (q, N(p-1)(1-\theta)/(N-1-\theta(p-1)))$ and $E \subset \Omega$ be a measurable set. Then, we have by Hölder's inequality

$\int_E |\nabla u_n|^q dx \leq (\int_E |\nabla u_n|^s dx)^{q/s} |E|^{1-q/s} \leq C|E|^{1-q/s} \to 0$ uniformly in n, as $|E| \to 0$.

From this and Lemma 6, we deduce that ∇u_n converges strongly to ∇u in $L^q(\Omega)$.

Now assume that $0 \leq \theta < \min\{(N-p+1)/(N-1), N/(N-1) - 1/(p-1), (p-r)/(p-1)\}$. Note that since ∇u_n converges to ∇u a.e. in Ω, to prove the strong convergence

$a(x, \nabla u_n)/(1+|u_n|)^{\theta(p-1)} \to a(x, \nabla u)/(1+|u|)^{\theta(p-1)}$ in $(L^1(\Omega))^N$,

it suffices, thanks to Vitali's Theorem, to show that for every measurable subset $E \subset \Omega$, $\int_E |a(x,\nabla u_n)/(1+|u_n|)^{\theta(p-1)}|dx$ converges to 0 uniformly in n, as $|E| \to 0$. Note that $p-1 < N(p-1)(1-\theta)/(N-1-\theta(p-1))$ by assumptions. For any $q \in (p-1, N(p-1)(1-\theta)/(N-1-\theta(p-1)))$, we deduce by Hölder's inequality

$\int_E |a(x,\nabla u_n)/(1+|u_n|)^{\theta(p-1)}|dx \leq \beta \int_E (j + |\nabla u_n|^{p-1})dx \leq \beta \|j\|_{p'} |E|^{1/p'} + \beta(\int_E |\nabla u_n|^q dx)^{(p-1)/q} |E|^{1-(p-1)/q} \to 0$ uniformly in n as $|E| \to 0$.

Lemma 7 There exists a subsequence of $\{u_n\}$ such that for all $k > 0$

$a(x, \nabla T_k(u_n))/(1+|T_k(u_n)|)^{\theta(p-1)} \to a(x, \nabla T_k(u))/(1+|T_k(u)|)^{\theta(p-1)}$ strongly in $(L^1(\Omega))^N$.

Proof. Let k be a positive number. It is well known that if a sequence of measurable functions $\{u_n\}$ with uniform boundedness in $L^s(\Omega)$ $(s > 1)$ converges in measure to u, then u_n converges strongly to u in $L^1(\Omega)$. First note that the sequence $\{a(x, \nabla T_k(u_n))/(1+|T_k(u_n)|)^{\theta(p-1)}\}$ is bounded in $L^{p'}(\Omega)$. Indeed, we have by (3) and Lemma 2,

$\int_\Omega |a(x, \nabla T_k(u_n))/(1 + |T_k(u_n)|)^{\theta(p-1)}|^{p'} dx \leq \beta\|j\|_{p'}^{p'} + \beta \int_\Omega |\nabla T_k(u_n)|^p/(1 + |T_k(u_n)|)^{\theta p} dx \leq C.$

Next, it suffices to show that there exists a subsequence of $\{u_n\}$ such that

$a(x, \nabla T_k(u_n))/(1 + |T_k(u_n)|)^{\theta(p-1)} \to a(x, \nabla T_k(u))/(1 + |T_k(u)|)^{\theta(p-1)}$ in measure.

Note that $u_n, u \in W^{1,q}(\Omega)$, where q is the same as in Proposition 4. The a.e. convergence of u_n to u and the fact that $\nabla u_n \to \nabla u$ in measure imply that $\nabla T_k(u_n) \to \nabla T_k(u)$ in measure.

Let s, t be positive numbers and write $\nabla A_u = a(x, \nabla u)/(1 + |u|)^{\theta(p-1)}$. Define

$E_n = \{|\nabla A_{T_k(u_n)} - \nabla A_{T_k(u)}| > s\}$,
$E_n^1 = \{|\nabla T_k(u_n)| > t\}$,
$E_n^2 = \{|\nabla T_k(u)| > t\}$,
$E_n^3 = E_n \cap \{|\nabla T_k(u_n)| \leq t\} \cap \{|\nabla T_k(u)| \leq t\}$.

Note that $E_n \subset E_n^1 \cup E_n^2 \cup E_n^3$ for each $n \geq 1$. Using the fact by Lemma 5 that the sequence $\{u_n\}$ and the function u are uniformly bounded in $W^{1,q}(\Omega)$, we obtain

$L^N(E_n^1) = \int_{\{|\nabla T_k(u_n)|>t\}} 1 dx \leq (1/t^q) \int_\Omega |\nabla T_k(u_n)|^q dx \leq C/t^q$,
$L^N(E_n^2) = \int_{\{|\nabla T_k(u)|>t\}} 1 dx \leq (1/t^q) \int_\Omega |\nabla T_k(u)|^q dx \leq C/t^q$.

We deduce that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$L^N(E_n^1) + L^N(E_n^2) < 2\varepsilon/3, \forall t \geq t_\varepsilon, \forall n \geq 1.$

Note that for $a \geq b \geq 0, \tau \geq 0$, we have the following inequality

$|(1 + a)^\tau - (1 + b)^\tau| \leq \tau|b - a|(1 + c)^{1+\tau} \leq \tau|b - a|$ for some $c \in [b, a]$.

We deduce from (5) and (3) that in E_n^3,

$s < |\nabla A_{T_k(u_n)} - \nabla A_{T_k(u)}| = |a(x, \nabla T_k(u_n))/(1 + |T_k(u_n)|)^{\theta(p-1)} - a(x, \nabla T_k(u))/(1 + |T_k(u)|)^{\theta(p-1)}| \leq \theta(p-1)|T_k(u_n) - T_k(u)| \cdot \beta(j + |\nabla T_k(u)|^{p-1}) + \{|\nabla T_k(u_n) - \nabla T_k(u)|^{p-1}, \text{ if } p \geq 2; |\nabla T_k(u_n) - \nabla T_k(u)|(1 + |\nabla T_k(u_n)| + |\nabla T_k(u)|)^{p-2}, \text{ if } p < 2\} \leq C_0 j|T_k(u_n) - T_k(u)| + C_0(1 + t^{p-1} + t^{p-2})(|T_k(u_n) - T_k(u)| + |\nabla T_k(u_n) - \nabla T_k(u)|),$

which leads to $E_n^3 \subset F_1 \cup F_2$, with

$F_1 = \{j|T_k(u_n) - T_k(u)| > s/(2C_0(1 + t^{p-1} + t^{p-2}))\}$,
$F_2 = \{|T_k(u_n) - T_k(u)| + |\nabla T_k(u_n) - \nabla T_k(u)| > s/(2C_0(1 + t^{p-1} + t^{p-2}))\}$.

In F_1, we have

$L^N(F_1) = \int_{F_1} 1dx \le (2C_0(1+t^{p-1}+t^{p-2})/s) \int_\Omega j|T_k(u_n) - T_k(u)|dx.$

By Lemma 3, we deduce that there exists $n_0 = n_0(s, C_0, \varepsilon)$ such that

$L^N(F_1) \le \varepsilon/6, \forall n \ge n_0.$

For F_2, note that $F_2 \subset F_3 \cup F_4$, with

$F_3 = \{|T_k(u_n) - T_k(u)| > s/(4C_0(1+t^{p-1}+t^{p-2}))\}$,
$F_4 = \{|\nabla T_k(u_n) - \nabla T_k(u)| > s/(4C_0(1+t^{p-1}+t^{p-2}))\}$.

Using the convergence in measure of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ and $T_k(u_n)$ to $T_k(u)$, for $t = t_\varepsilon$, we obtain the existence of $n_1 = n_1(s, \varepsilon) \ge 1$ such that

$L^N(F_2) \le L^N(F_3) + L^N(F_4) < \varepsilon/6, \forall n \ge n_1.$

Combining (35), (36) and (37), we obtain

$L^N(\{|\nabla A_{T_k(u_n)} - \nabla A_{T_k(u)}| > s\}) < \varepsilon, \forall n \ge \max\{n_0, n_1\}.$

Hence the sequence $\{\nabla A_{T_k(u_n)}\}$ converges in measure to $\nabla A_{T_k(u)}$ and the lemma follows.

3 Proof of the main result

Now we have gathered all the lemmas needed to prove the existence of an entropy solution to the obstacle problem associated with (f, ψ, g). In this part, let f_n be a sequence of smooth functions converging strongly to f in $L^1(\Omega)$, with $\|f_n\|_1 \le \|f\|_1 + 1$. We consider the sequence of approximated obstacle problems associated with (f_n, ψ, g).

Proof of Theorem 1 Let $v \in K_{g,\psi} \cap L^\infty(\Omega)$. Taking v as a test function in (7) associated with (f_n, ψ, g), we get

$\int_\Omega a(x, \nabla u_n)/(1+|u_n|)^{\theta(p-1)} \cdot \nabla(T_t(u_n - v))dx + \int_\Omega b|u_n|^{r-2}u_n T_t(u_n - v)dx \le \int_\Omega f_n T_t(u_n - v)dx.$

Since $\{|u_n - v| < t\} \subset \{|u_n| < s\}$ with $s = t + \|v\|_\infty$, the previous inequality can be written as

$\int_\Omega \chi_n \nabla A_{T_s(u_n)} \cdot \nabla v dx \ge -\int_\Omega f_n T_t(u_n - v)dx + \int_\Omega b|u_n|^{r-2}u_n T_t(u_n - v)dx + \int_\Omega \chi_n \nabla A_{T_s(u_n)} \cdot \nabla T_s(u_n)dx,$

where $\chi_n = \chi_{|u_n - v|<t}$ and $\nabla A_u = a(x, \nabla u)/(1+|u|)^{\theta(p-1)}$. It is clear that $\chi_n \rightharpoonup \chi$ weakly* in $L^\infty(\Omega)$. Moreover, χ_n converges a.e. to $\chi_{|u-v|<t}$ in Ω $\{|u-v| = t\}$. It follows that

$\chi = \{1, \text{ in } \{|u-v| < t\}, 0, \text{ in } \{|u-v| > t\}\}.$

Note that we have $L^N(\{|u-v| = t\}) = 0$ for a.e. $t \in (0, \infty)$. So there exists a measurable set $O \subset (0, \infty)$ such that $L^N(\{|u-v| = t\}) = 0$ for all $t \in (0, \infty)$ O.

Assume that $t \in (0,\infty) \quad O$. Then χ_n converges weakly* in $L^\infty(\Omega)$ and a.e. in Ω to $\chi = \chi_{|u-v|<t}$. Since $\nabla T_s(u_n)$ converges a.e. to $\nabla T_s(u)$ in Ω (Proposition 4), we obtain by Fatou's Lemma

$\liminf_{n\to\infty} \int_\Omega \chi_n \nabla A_{T_s(u_n)} \cdot \nabla T_s(u_n) dx \geq \int_\Omega \chi \nabla A_{T_s(u)} \cdot \nabla T_s(u) dx.$

Using the strong convergence of $\nabla A_{T_s(u_n)}$ to $\nabla A_{T_s(u)}$ in $L^1(\Omega)$ (Lemma 7) and the weak* convergence of χ_n to χ in $L^\infty(\Omega)$, we obtain

$\lim_{n\to\infty} \int_\Omega \chi_n \nabla A_{T_s(u_n)} \cdot \nabla v dx = \int_\Omega \chi \nabla A_{T_s(u)} \cdot \nabla v dx.$

Moreover, due to the strong convergence of f_n to f and $|u_n|^{r-2}u_n$ to $|u|^{r-2}u$ (by $r-1 < q^*$ and the boundedness of $\|u_n\|_{q^*}$) in $L^1(\Omega)$, and the weak* convergence of $T_t(u_n - v)$ to $T_t(u-v)$ in $L^\infty(\Omega)$, by passing to the limit in (38) and taking into account (39)-(40), we obtain

$\int_\Omega \chi \nabla A_{T_s(u)} \cdot \nabla v dx - \int_\Omega \chi \nabla A_{T_s(u)} \cdot \nabla T_s(u) dx \geq -\int_\Omega f T_t(u-v) dx + \int_\Omega b|u|^{r-2}u T_t(u-v) dx,$

which can be written as

$\int_{\{|v-u|\leq t\}} \chi \nabla A_{T_s(u)} \cdot (\nabla v - \nabla u) dx \geq -\int_\Omega f T_t(u-v) dx + \int_\Omega b|u|^{r-2}u T_t(u-v) dx,$

or since $\chi = \chi_{|u-v|<t}$ and $\nabla(T_t(u-v)) = \chi_{|u-v|<t} \nabla(u-v)$

$\int_\Omega \nabla A_u \cdot \nabla T_t(u-v) dx + \int_\Omega b|u|^{r-2}u T_t(u-v) dx \leq \int_\Omega f T_t(u-v) dx, \forall t \in (0,\infty) \quad O.$

For $t \in O$, we know that there exists a sequence $\{t_k\}$ of numbers in $(0,\infty) \quad O$ such that $t_k \to t$ due to $|O| = 0$. Therefore, we have

$\int_\Omega \nabla A_u \cdot \nabla T_{t_k}(u-v) dx + \int_\Omega b|u|^{r-2}u T_{t_k}(u-v) dx \leq \int_\Omega f T_{t_k}(u-v) dx.$

Since $\nabla(u-v) = 0$ a.e. in $\{|u-v| = t\}$, the left-hand side of (41) can be written as

$\int_\Omega \nabla A_u \cdot \nabla T_{t_k}(u-v) dx = \int_{\Omega \quad \{|u-v|=t\}} \chi_{|u-v|<t_k} \nabla A_u \cdot \nabla(u-v) dx.$

The sequence $\chi_{|u-v|<t_k}$ converges to $\chi_{|u-v|<t}$ a.e. in $\Omega \quad \{|u-v| = t\}$ and therefore converges weakly* in $L^\infty(\Omega \quad \{|u-v| = t\})$. We obtain

$\lim_{k\to\infty} \int_\Omega \nabla A_u \cdot \nabla T_{t_k}(u-v) dx = \int_{\Omega \quad \{|u-v|=t\}} \chi_{|u-v|<t} \nabla A_u \cdot \nabla(u-v) dx = \int_\Omega \nabla A_u \cdot \nabla T_t(u-v) dx.$

For the right-hand side of (41), we have

$|\int_\Omega f T_{t_k}(u-v) dx - \int_\Omega f T_t(u-v) dx| \leq |t_k - t| \cdot \|f\|_1 \to 0$ as $k \to \infty$.

Similarly, we have

$|\int_\Omega |u|^{r-2}u T_{t_k}(u-v) dx - \int_\Omega |u|^{r-2}u T_t(u-v) dx| \leq |t_k - t| \cdot \||u|^{r-1}\|_1 \to 0$ as $k \to \infty$.

It follows from (41)-(44) that we have the inequality

$\int_\Omega \nabla A_u \cdot \nabla T_t(u-v) dx + \int_\Omega b|u|^{r-2}u T_t(u-v) dx \leq \int_\Omega f T_t(u-v) dx, \forall t \in (0,\infty).$

Hence, u is an entropy solution of the obstacle problem associated with (f, ψ, g). The dependence of the entropy solution on the data $f \in L^1(\Omega)$ is guaranteed by Proposition 4.

4.1 Availability of data and material

Not applicable.

4.2 Competing interests

The author declares that he has no competing interests.

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4.4 Authors' contributions

This paper was completed by JZ independently.

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