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# On the Analytical Solution of Accelerating Structure

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## Abstract

This paper discusses analytical solution problems in accelerating structures, focusing on two solution methods: field-based and equivalent circuit-based approaches.

## Full Text

## Preamble

**On the Analytical Solution of Accelerating Structures**
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## Abstract

This paper discusses analytical solutions for electromagnetic fields in accelerating structures, focusing on field and circuit methods for analyzing both traveling-wave and standing-wave accelerating structures. Key results are presented.

**Keywords:** Accelerating structure, Analytical solution, Dispersion relation.

## 1 Introduction

The accelerating structure represents the core component of any accelerator system [1, 2, 3, 4, 5, 6]. These structures typically operate in the TM mode and fall into two fundamental categories: traveling-wave and standing-wave accelerating structures. Proton accelerators predominantly employ standing-wave structures, whereas electron accelerators generally utilize traveling-wave configurations. For electron accelerators, several standard frequency bands are widely used in accelerator laboratories worldwide: L-band (1300 MHz), S-band

(2856 MHz), C-band (5712 MHz), and X-band (11.4 GHz). Figure 1 [Figure 1: see original paper] illustrates the schematic model of a disk-loaded structure.

A primary characteristic of electromagnetic waves is their dispersion relation, which describes the functional relationship between wave frequency and wave number: $f(\omega, k) = 0$. The phase velocity is given by $v_p = \omega/k$, while the group velocity is $v_g = d\omega/dk$. Waves with phase velocity exceeding the speed of light are termed fast waves, whereas those with phase velocity less than the speed of light are called slow waves.

## 2 Time-Harmonic Factor of Traveling-Wave Fields

Accelerating structures are classified into traveling-wave and standing-wave types, and analytical methods for solving their electromagnetic fields similarly fall into two approaches: analytical and numerical techniques. Analytical methods are restricted to relatively simple geometries and belong to closed-form electromagnetic field theory. However, with the rapid development of computers and numerical methods, electromagnetic simulation software such as MAFIA, Superfish, CST, and HFSS has become available, making simulation-based research routine and causing analytical methods to be gradually overlooked. This paper reintroduces analytical methods for solving electromagnetic fields in accelerating structures.

In cylindrical coordinates $(r, \theta, z)$, the time-harmonic factor for traveling-wave fields in accelerating structures takes the form $\exp(i(\omega t - kz))$. Standing waves can be decomposed into two counter-propagating traveling waves. Due to the linear superposition principle of Maxwell' s equations, we focus our discussion on the traveling-wave case.

## 3 The Basic Wave Equations

In accelerating structures, the source-free wave equations can be expressed as:

$$\mathbf{E} = -$$

Expanding these equations yields:

$$H_r = - \quad H_\theta = - \quad H_z = -$$

## 4 Solving the Wave Equations from Longitudinal Field Components

Expressing transverse fields in terms of longitudinal components gives:

$$E_r = (c^2\mu\varepsilon - k_z \quad E_\theta = (c^2\mu\varepsilon - k_z \quad H_r = (c^2\mu\varepsilon - k_z \quad H_\theta = (c^2\mu\varepsilon - k_z$$

The longitudinal field components satisfy Helmholtz equations:

$$(\nabla^2 + c^2\mu\varepsilon)E_z = 0, \quad (\nabla^2 + c^2\mu\varepsilon)H_z = 0.$$

Thus, we first solve the longitudinal field equations and subsequently derive the transverse fields from these solutions.

## 5 The Eigenmode Equation

For the Helmholtz equation governing longitudinal field components, we decompose the operator as $\nabla^2 \to \nabla_\perp^2 + d^2/dz^2$. Applying the traveling-wave condition yields the partial differential equation:

$$((cid:53)^2 c^2\mu\varepsilon - k_z^2)f = 0$$

where $f$ represents the longitudinal field component. The eigenmode equation becomes:

$$(\nabla_\perp^2 + \beta_n^2)f = 0.$$

In cylindrical coordinates, the TM eigenmode equation is:

$$\partial_\theta^2 + \beta_n^2)f_n = 0$$

For $\mathrm{TM}_{0n}$ modes, where $\partial^2/\partial\theta^2 = 0$, the equation simplifies to Bessel's equation with solution:

$$f_n = AJ_0(\beta_n r)$$

where the conventional fundamental mode is $\mathrm{TM}_{01}$.

## 6 The Equivalent Circuit and Dispersion Relation

The equivalent circuit method provides a straightforward tool for determining the principal characteristics of accelerating structures. The fundamental approach expands the electric field as $\mathbf{E}(r_\perp)V(z,\omega)$ and the magnetic field as $\mathbf{H}(r_\perp)I(z,\omega)Z$, where $Z$ represents the wave impedance. This method enables derivation of the structure’ s dispersion relation.

Physically, we model the accelerating structure as a series of coupled resonant cavities. The equivalent circuit for a traveling-wave accelerating structure is shown in Figure 2 [Figure 2: see original paper]. From this circuit, we obtain the general dispersion relation:

$$\omega^2 = \omega_0^2(1 - \kappa\cos(\theta))$$

where $\omega_0$ and $\theta$ are the resonant frequency and phase shift, respectively, and $\kappa$ is the coupling coefficient. The coupling coefficient can be roughly estimated as:

$$\kappa \approx \frac{a}{L}$$

where $a$ is the bore radius and $L$ is the cavity length. The group velocity for the traveling-wave structure is:

$$v_g = \frac{\kappa\sin(\theta)}{\sqrt{1 - \kappa\cos(\theta)}}$$

Using typical parameters from the SLAC S-band structure (operating frequency 2856 MHz), the inter-cell coupling factor is approximately 0.15. The corresponding dispersion relation curve and group velocity are shown in Figure 3 [Figure 3: see original paper] and Figure 4 [Figure 4: see original paper], respectively.

The coupling factor can be determined experimentally or through field analysis. Accelerating structures can operate in either traveling-wave or backward-wave modes, with the operating mode determined by the working point on the dispersion curve. In the traveling-wave region, phase and group velocities share the same direction, whereas in the backward-wave region they are opposite.

For standing-wave structures, the equivalent circuit is shown in Figure 5 [Figure 5: see original paper]. The derived dispersion relation is:

$$\omega^2 = \omega_0^2\frac{1 - \kappa\cos(\theta)}{1 + \kappa\cos(\theta)}$$

In standing-wave structures, frequency is independent of wave number, resulting in zero group velocity. The filling time corresponds to the cavity field buildup time. Since $\kappa$ is small, the standing-wave dispersion relation can be approximated as:

$$\omega^2 \approx \omega_0^2(1 - \kappa\cos(\theta))$$

While both traveling-wave and standing-wave structures can be described by similar dispersion relations, the underlying physics differs substantially. Although analytical results provide simple physical insight, numerical and experimental studies remain essential for further investigation.

## 7 Discussion

With the development and maturation of computational and numerical methods, numerical experiments have become an important frontier in physics research. Classical macroscopic electromagnetic theory, governed by Maxwell' s partial differential equations, has reached a high level of simulation maturity. Structures with analytical solutions represent only a tiny fraction of all possible configurations. Nevertheless, familiarity with analytical methods proves invaluable for numerical experimentation.

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