Gauge Potential Decomposition Theory and Global Topological Problems

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Date: 2018-06-02T00:00:00+00:00

Abstract

Utilizing the concept proposed by Duan Yishi that gauge potentials are decomposable and possess internal structure, we employ geometric algebra methods to decompose the SO(n) group using unit vector fields, presenting a general form and discussing the properties of this decomposition. This yields the decomposition forms for the SU(2) and U(1) groups using unit vectors, which are precisely the results given by the famous physicist Faddeev in 1999. Using the general form of the SO(n) group gauge potential decomposition, we discuss the local topological structure of the Gauss-Bonnet-Chern density, whose global topological structure is exactly the Gauss-Bonnet-Chern theorem; from this topological structure, the Morse theory form of the Euler-Poincaré characteristic can be easily obtained. By studying the -1/2 Bose-Einstein condensate using the SU(2) group gauge potential decomposition, we derive a new circulation condition, which is also a generalization of the Mermin-Ho relation. Finally, using the relationship between the torsion tensor in three-dimensional Riemannian geometry and U(1) gauge theory discovered by Duan Yishi, we investigate the relationship between dislocation lines and the linking number using U(1) gauge potential decomposition.

Full Text

Decomposition Theory of Gauge Potential and Global Topology Problems

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Abstract

Using the idea of gauge potential decomposability and internal structure proposed by Duan Yi-Shi, we decompose the gauge potential of the SO(n) group using unit vector fields through geometric algebra methods, obtaining a general decomposition form and discussing its properties. This paper presents the decomposition forms for the SU(2) and U(1) groups using unit vector fields, which coincide exactly with the results given by the renowned physicist Faddeev in 1999. Employing the general form of SO(n) gauge potential decomposition, we examine the local topological structure of the Gauss-Bonnet-Chern density, whose global topological structure precisely yields the Gauss-Bonnet-Chern theorem. From this topological structure, the Morse-theoretic form of the Euler-Poincaré characteristic number is readily obtained. Using SU(2) gauge potential decomposition to study spin-1/2 Bose-Einstein condensates, we derive a new circulation condition that generalizes the Mermin-Ho relation. Finally, utilizing the relationship between the torsion tensor in three-dimensional Riemannian geometry and U(1) gauge theory discovered by Duan Yi-Shi, we investigate the relationship between dislocation lines and link numbers through U(1) gauge potential decomposition.

Keywords: Decomposition of gauge potential; Global topology; Link number
PACS: 11.15-q; 02.40.Ma; 02.10.kn

Gauge field theory serves as the foundation for studying the four fundamental interactions. Electromagnetic interaction is a U(1) gauge theory, strong interaction is an SU(3) gauge theory, and the electroweak unification (weak and electromagnetic interactions) is an SU(2)×U(1) gauge theory, where the weak field mass is resolved through the Higgs mechanism. Duan Yi-Shi' s research demonstrates that gravitational interaction is also a GL(4) gauge theory. While the quantization of the first three interaction fields can be achieved through path integral quantization methods, the quantization of the gravitational field remains incompletely resolved.

Gauge field theory was established by Yang and Mills in 1954, and its mathematical structure precisely corresponds to the fiber bundle theory developed by Chern in the 1950s. The development of particle physics is inseparable from gauge theory, which has played an irreplaceable role in physicists' investigations of the microscopic structure of matter and interactions. In the history of physics, Newton' s classical mechanics, Einstein' s relativity (including general relativity), Maxwell' s electromagnetic theory, and Yang-Mills gauge theory are all epoch-making achievements, with gauge theory being the key to unlocking the structure of microscopic matter.

When Duan Yi-Shi studied the generally covariant Dirac equation in the 1950s, he discovered that the spin connection could be expressed using a semi-metric, implying that gauge potentials could be represented by other fields. In the 1970s, Duan Yi-Shi and Ge Mo-lin found during their research on magnetic monopoles that the SU(2) gauge potential could be expressed using unit vector fields, which marks the origin of gauge potential decomposition. The decomposability of gauge potentials represents a significant academic idea and method in gauge field research.

Over the past three decades, foreign physicists have also recognized that gauge potentials are decomposable. The Wang Fan team in China has applied the idea of gauge potential decomposition to study the proton spin structure problem, obtaining excellent results. The decomposition of spin connections for SO(n) was research conducted by Duan Yi-Shi and his students in the 1990s. The decomposition of spin connections using unit vector fields can be used to study the topological structure of the Gauss-Bonnet-Chern density, whose globality provides a complete proof of the Gauss-Bonnet-Chern theorem and readily yields the Morse-theoretic representation of the Euler-Poincaré characteristic number, revealing the connections between them.

Gauge potential decomposition theory is an effective method for studying topological problems in physics. We present several examples of topological problems across five chapters. From current research on gauge fields, physicists' understanding remains incomplete. For instance, what constitutes a gauge transformation? How should we understand the contribution of gauge fields to nucleon spin? Research from Professor Wang Fan' s group demonstrates how to select gauge potentials that represent physical reality in specific problems—questions that still lack final answers.

II. Gauge Potential Decomposition Theory

In gauge theory, the connection 1-form (gauge potential), curvature tensor 2-form (gauge field strength), and sections (fundamental fields) satisfy fundamental transformation properties. When the connection 1-form undergoes a gauge transformation, the curvature tensor 2-form transforms covariantly, and the fundamental fields and their covariant derivatives also satisfy covariant transformation laws. These constitute the foundation of gauge theory.

A gauge potential A satisfying gauge transformations can always be decomposed into two parts, each required to satisfy gauge transformation and vector covariant transformation laws respectively. The sum of these parts still satisfies the gauge transformation law. Furthermore, these components can be constructed from fundamental fields, which forms the basis of gauge potential decomposition. In specific physical applications, the challenge lies in selecting the physically relevant gauge field.

Spin Connection Decomposition Using Unit Vector Fields

Let M be a compact n-dimensional Riemannian manifold, and let ϕ be a smooth section on a vector bundle over M. The zero points of ϕ correspond to singular points of the unit vector field with natural constraints, forming a section on the sphere bundle over M.

In gauge theory, the vector bundle on a Riemannian manifold is the principal bundle $GM_p(\pi)$, where x^μ are local coordinates on the base manifold. Its connection is called the spin connection, denoted by ω^{ab}, with curvature tensor potential form R^{ab}. The covariant derivative of a unit vector field is defined with a natural normalization condition.

Let γ^a be n-dimensional Dirac matrices satisfying the Clifford algebra. The generators of the group are given by $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$. In geometric algebra theory, γ^a serve as basis vectors, and the unit vector field n^a can be represented in matrix form as a vector in geometric algebra. The spin connection and curvature tensor can be expressed in terms of these geometric algebra elements.

Let T_r be an r-vector in geometric algebra. It can be shown that the covariant derivative 1-form satisfies specific relations, leading to the general decomposition of the spin connection using unit vector fields:

$$\omega^{ab} = n^a Dn^b - n^b Dn^a + D\chi^{ab}$$

where we have introduced the notation $D\chi^{ab} = B^{ab}$. These two equations constitute the general form of spin connection decomposition using unit vector fields, which possesses three important properties:

1. **Global Nature**: If $\{U, V, ...\}$ is an open cover of M with transition functions S_{UV} satisfying $S_{UV} \neq 0$ on $U \cap V$, then for unit vector fields on any open domain, the spin connection decomposition satisfies the relation $S_{UV} \omega S_{UV}^{-1} = \omega'$. This demonstrates that when the decomposition holds on open neighborhood U, it necessarily holds on open neighborhood V, meeting the fundamental requirement for connections on principal bundles.

2. **Independence from Specific Vector Field Choice**: For another vector field n'^a independent of n^a, simple calculations show that the general decomposition does not depend on the specific choice of unit vector field.

3. **Truncation Problem**: For any element B^{ab} in the SO(n) Lie algebra, we can construct a decomposition component $a^{ab} = SB^{ab}S^{-1}$. Since a^{ab} is arbitrary, B^{ab} also has arbitrariness while satisfying gauge transformations. This property is crucial in applications because while the gauge potential in the general decomposition form is cyclic, this property shows that B^{ab} can be arbitrary, providing flexibility in selecting gauge potentials for specific problems.

IV. Decomposition of U(1) and SU(2) Gauge Potentials

In practical applications, U(1) and SU(2) gauge potentials are most commonly encountered, making their decomposition forms particularly important.

1. General Decomposition Form of U(1) Gauge Field

Let T be the generator of the U(1) group, which relates to the SO(2) group generator through $T = \epsilon^{ab} T^{ab}$. The U(1) gauge potential is defined as $A_\mu = A_\mu^{ab} \epsilon^{ab}$, which is dual to the SO(2) gauge potential. It can be shown that $B^{ab} = J\epsilon^{ab}$, leading to the general decomposition forms for U(1) gauge potential and field strength:

$$A_\mu = \epsilon^{ab} n^a \partial_\mu n^b + B_\mu$$
$$F_{\mu\nu} = 2\epsilon^{ab} D_\mu n^a D_\nu n^b + H_{\mu\nu}$$

When n^a is a gauge-parallel unit vector field satisfying $D_\mu n^a = 0$, the decomposition simplifies to $A_\mu = B_\mu$ and $F_{\mu\nu} = H_{\mu\nu}$. It can be proven that A_μ and $A_\mu^{||}$ differ by a gauge transformation.

2. General Decomposition Form of SU(2) Gauge Potential

The SO(3) and SU(2) groups share the same local structure, meaning they have identical Lie algebra structures. Let T^a be the SU(2) group generators, related to the SO(3) group generators by $T^a = \frac{1}{2}\sigma^a$. The SU(2) gauge potential relates to the SO(3) gauge potential through $A_\mu = A_\mu^a T^a$, and the gauge field strengths are similarly related.

After straightforward calculation, we obtain the general decomposition forms for SO(3) and SU(2) gauge potentials and field strengths:

$$A_\mu^a = \epsilon^{abc} n^b \partial_\mu n^c + B_\mu^a$$
$$F_{\mu\nu}^a = \epsilon^{abc}(D_\mu n^b D_\nu n^c + n^b F_{\mu\nu}^c) + H_{\mu\nu}^a$$

These equations (53) and (54) are precisely the results given by the renowned physicist Faddeev et al. in 1999. When n^a is a gauge-parallel unit vector field, these expressions simplify further. Under the axial gauge condition $n^a A_\mu^a = 0$, we obtain simplified forms that are particularly useful in applications.

V. Applications to Topological Problems

1. Topological Structure of Gauss-Bonnet-Chern Density and the Gauss-Bonnet-Chern Theorem

In the general decomposition of SO(N) group spin connection (35), the connection B^{ab} can be arbitrary. Here we choose the flat connection $B^{ab} = 0$, which transforms (35) into a simplified form. For a flat connection, $F^{ab}(B) = 0$, and from (36) we obtain expressions for the field strength in terms of the unit vector field derivatives.

For a compact oriented Riemannian manifold M, there exists an independent n-form Λ that is closed. Chern proved that the $(n-1)$-form Ω constructed from the curvature form F^{ab} is a Chern form. Using the Bianchi identity $DF^{ab} = 0$, one can show that $d\Lambda = 0$, meaning Λ is independent of the spin connection and depends only on the cohomology group, representing a topological invariant known as the Euler-Poincaré characteristic class.

The famous Gauss-Bonnet-Chern theorem states:

$$\int_M \Lambda = \chi(M)$$

where $\chi(M)$ is the Euler-Poincaré characteristic number of manifold M. Substituting the field strength obtained with $B^{ab} = 0$ into the expression for Λ and simplifying using geometric algebra, we can construct the flat connection in a simple matrix form that represents a local spin representation of the SO(n) group.

Using the ϕ-mapping method, the exterior derivative of the $(n-1)$-form Ω pulled back to M yields the Gauss-Bonnet-Chern density on M. These expressions reveal that only when $\phi = 0$ are there N isolated zero points on M. Let the zero points be at z_i $(i = 1, ..., N)$, where β_i is the Hopf index (a positive integer) and η_i is the Brouwer degree of the i-th zero point.

The topological structure of the GBC density on M can be expressed as:

$$\Lambda = \sum_{i=1}^{N} \beta_i \eta_i \delta(x - z_i) d^n x$$

This shows that the GBC density can be expressed in terms of the Brouwer degrees and Hopf indices of the zero points. Integrating over M gives:

$$\int_M \Lambda = \sum_{i=1}^{N} \beta_i \eta_i = \chi(M)$$

Let f be a Morse function on manifold M. A point p is a critical point of f if $\partial_\mu f(p) = 0$. The Hessian matrix at p is defined as $H_{\mu\nu}(p) = \partial_\mu \partial_\nu f(p)$. The zero points of the gradient vector field $\phi^a = \partial^a f$ correspond to the critical points of f. In Morse theory, for a Morse function, the Hopf index $\beta_i = 1$ for each critical point, leading to the Morse-theoretic result of the Gauss-Bonnet-Chern theorem:

$$\chi(M) = \sum_{i=1}^{N} (-1)^{\lambda_i}$$

where λ_i is the Morse index at the i-th critical point. Thus, we derive the Morse-theoretic result of the Gauss-Bonnet-Chern theorem from the topological structure of the GBC density, demonstrating that the global topological structure is determined by the local topological structure of the zero points.

2. SU(2) Gauge Field Decomposition and Circulation Condition in Bose-Einstein Condensates

Bose-Einstein condensates are important systems in condensed matter physics, and understanding their global properties is crucial. We have discovered a circulation condition related to topology. Considering the projection of the gauge field strength onto the unit vector field in SU(2) gauge field decomposition, we obtain the ' t Hooft tensor.

The order parameter for a general spinor Bose condensate is chosen as $\Psi = \phi\zeta$, where ψ is the field operator, ϕ is a scalar field, and ζ is a normalized spinor field. For a two-component condensate of spin-1/2 Bose gases, the order parameter is a spinor. Although true SU(2) symmetry does not exist in the condensate, when the scattering lengths of the two components are equal, the system possesses approximate SU(2) symmetry. We introduce an SU(2) gauge field A_μ^a $(a = 0, 1, 2, 3)$ and impose the constraint $A_\mu^0 = 0$ to eliminate unphysical gauge degrees of freedom.

To relate the spin gauge potential to the superfluid velocity, we consider the projection of the gauge field B_μ^a onto the mean spin direction $N^a = \zeta^\dagger \sigma^a \zeta$. The spatial components of this projection remain invariant under U(1) rotation about the mean spin direction. The ' t Hooft tensor relates to the superfluid velocity as:

$$v_\mu = \frac{1}{2m}(N^a D_\mu N^a)$$

where v_μ is the superfluid velocity. This relation reveals a new structure for the superfluid vorticity that depends not only on the mean spin direction but also on the field distribution, representing a generalization of the Mermin-Ho relation.

Considering the integral over a surface Σ with boundary C, we find that the integral $\oint_C v \cdot dl$ is a topological integer. Therefore, we obtain the circulation condition for spin-1/2 condensates:

$$\oint_C \mathbf{v} \cdot d\mathbf{l} = \frac{h}{m}\mathcal{W}$$

where $\mathcal{W}$ is a topological invariant. This new topological constraint differs significantly from the circulation condition in single-component Bose condensates. The relation also shows how the mean spin direction contributes to the circulation, making quantization possible only for special configurations of the N^a field.

3. Torsion Structure on Riemann-Cartan Manifold and Knots

Knots are ubiquitous phenomena. In the late 20th century, physicists discovered them in field theory, and we have found that dislocation lines in solids may form link numbers. By projecting the torsion tensor onto vector fields to introduce U(1) gauge potentials, we can use gauge potential decomposition to see how dislocation line geometries in solids may form link numbers.

Let U_3 be a 3-dimensional Riemann-Cartan spacetime manifold with Vierbein field e^A_μ and spin connection ω^{AB}_μ. The torsion tensor is defined as:

$$T^\rho_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}$$

where Γ is the affine connection. If N^A is a gauge-parallel unit vector field satisfying $D_\mu N^A = 0$, then the projection of the torsion tensor onto N^A yields a U(1) gauge potential:

$$A_\mu = N_A T^A_{\mu\nu} dx^\nu$$

with the torsion tensor acting as the U(1) gauge field strength. For any given surface Σ, we can define the integral:

$$\Phi = \int_\Sigma T_{\mu\nu} d\sigma^{\mu\nu}$$

which represents the defect flux through the 2D surface Σ.

The projected torsion resembles a U(1) structure. In three-dimensional Riemann-Cartan space, this structure implies the existence of a Chern-Simons action:

$$S_{CS} = \frac{k}{4\pi} \int_M A \wedge dA$$

which is a Chern-Simons link invariant. For manifold M, gauge-invariant and metric-independent quantities should be topological invariants. For the Abelian case, S. Albeverio provided a rigorous mathematical proof.

Let C be an oriented closed curve in $\mathbb{R}^3$. For a general Lie group G, Duan Yi-Shi and Ge Mo-lin (1974) and Wilson (1974) introduced a parallel transport operator along C:

$$W_R(C) = \mathrm{Tr}\, P \exp\left(i \oint_C A_\mu^a T^a dx^\mu \right)$$

which represents the holonomy group element along C. For our Abelian case involving the U(1) group representation, let L be a link in M consisting of r oriented, non-intersecting knots C_i ($i = 1, ..., r$). The vacuum expectation value:

$$\langle W(L) \rangle = \frac{1}{Z} \int \mathcal{D}A\, e^{iS_{CS}} \prod_{i=1}^{r} W_{R_i}(C_i)$$

yields the link number through curve δ-functions and path integration:

$$\langle W(L) \rangle = \exp\left(2\pi i \sum_{i<j} \mathrm{Link}(C_i, C_j) \right)$$

where the link number is defined as:

$$\mathrm{Link}(C_i, C_j) = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot (d\mathbf{r}_i \times d\mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

For closed curves C_i and C_j, the link number is an integer and a topological invariant.

To provide further interpretation involving torsion loops, consider a surface Σ in $\mathbb{R}^3$ pierced by Wilson lines, creating intersection points. On this surface, the U(1) gauge transformation corresponds to SO(2) gauge transformation, with the relation:

$$A_\mu^{ab} = \lambda \epsilon^{ab} A_\mu$$

where λ is a dimension-dependent constant. Substituting into the flux integral yields:

$$\Phi = \lambda \int_\Sigma T_{\mu\nu}^{ab} d\sigma^{\mu\nu}$$

where $T_{\mu\nu}^{ab}$ acts as the SO(2) gauge field strength. Using the gauge potential decomposition (11) and (12), we obtain:

$$T_{\mu\nu}^{ab} = \epsilon^{ab} \epsilon^{cde} n^c \partial_\mu n^d \partial_\nu n^e$$

Applying the ϕ-mapping method, let the internal coordinates of surface Σ be (u^1, u^2) with Jacobian J. Assuming the vector field ϕ^a has r isolated zeros at $x = z_k$ $(k = 1, ..., r)$, we obtain:

$$\Phi = \lambda \sum_{k=1}^{r} \beta_k \eta_k \mathrm{sgn}(J_k)$$

where η_k is the Brouwer degree and β_k is the Hopf index of the mapping at the k-th zero. Therefore, the defect flux can be quantized by the wrapping number:

$$\Phi = \lambda \sum_{k=1}^{r} \beta_k \eta_k$$

which is a topological quantity. For dislocations in continuous media, λ is the lattice constant; for spacetime, λ might be the Planck length.

From the above discussion, Wilson loops (knots) may represent defect lines. The existence of SO(n) gauge potential decomposition in general form, obtained using geometric algebra, provides great convenience for discussing the topological properties of Gauss-Bonnet-Chern density. The zero points of vector fields precisely reflect the topological properties of space, and through gauge potential decomposition using vector fields, their topological properties are intimately connected, easily yielding a complete proof of the famous Gauss-Bonnet-Chern theorem and the Morse-theoretic form of the Euler-Poincaré characteristic number, demonstrating the unity of their zero-point topological properties.

The other two examples also illustrate the remarkable utility of the gauge potential decomposition idea. In fact, Wang Fan' s group has essentially clarified the contribution of gluon fields to nucleon spin composition using this method. Therefore, the idea of gauge potential decomposability represents a highlight in gauge field research, first proposed by Chinese scientists.

References

1. C.N. Yang and R. Mills, Phys. Rev. Lett. 1954, 96:191; arXiv:hep-ph/9812242V1
2. Xi-Guo Lee, Nöether' s Theorem and Duan Yi-Shi' s Generalized Covariant Conservation Law, Sci. Sin. G, 2018 (same period as this paper)
3. Albert S. Schwarz, *Topology For Physicists* (Springer-Verlag Press, 1994); C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic Press, London, 1983)
4. Duan Yi-Shi, The Generalized Covariant Equation of the Field of an Arbitrary Spin Basic Particle, Journal of Lanzhou University, 1958.1:29-34
5. Y.M. Cho, Phys. Rev. D, Restricted Gauge Theory, 1980, 21:1080-1090; Phys. Rev. Lett. Glueball Spectrum in Extended Quantum Chromodynamics, 1981, 46:302-309; Faddeev L. and Niemi Antti J., Partially Dual

Variables in SU(2) Yang-Mills Theory, Phys. Rev. Lett. 1999, 82:1624-1627
6. Xiang-Song Chen, Xiao-Fu Lu, Wei-Min Sun, Fan Wang, and T. Goldman, Phys. Rev. Lett. Spin and Orbital Angular Momentum in Gauge Theories: Nucleon Spin Structure and Multipole Radiation Revisited, 2008, 100:232002-232006
7. Y.S. Duan, M.L. Ge, SU(2) Gauge Theory and Electrodynamics with N Magnetic Monopoles, Sci. Sinica, 1979, 9(11):1072-1081
8. C. Doran, D. Hestenes, F. Sommen and N.V. Acker, J. Math. Phys. 1993, 34:3642
9. Yi-Shi Duan, Xi-Guo Lee, General Decomposition Theory of Spin Connections, Topological Structure of Gauss-Bonnet-Chern Density and Morse Theory, Helv. Phys. Acta, 1995, 68:513
10. Duan Yi-Shi, Li Xi-Guo, General Decomposition Theory of Spin Connection, Topological Structure in Gauss-Bonnet-Chern Density, Commun. Theor. Phys. 1998, 29:237
11. Li Xi-Guo, Duan Yi-Shi, Song Jian-Jun, General Decomposition Theory of SO(n) Gauge Potential, CPC, 2001, 25:373
12. Duan Yi-Shi, X.H. Meng, J. Math. Phys. 1993, 34:1149
13. S.S. Chern, Ann. Math. A Simple Intrinsic Proof of the Gauss-Bonnet Formula for Closed Riemannian Manifolds, 45, (1944) 747-752
14. Xi-Guo Lee, General Decomposition Theory of Gauge Potential, Gauss-Bonnet-Chern Theorem and Inner Structure of Torsion, Doctoral Dissertation, 1994, Lanzhou University
15. Duan Y.-S., Fu L.-B., The General Decomposition Theory of SU(2) Gauge Potential: Topological Structure and Bifurcation of SU(2) Chern Density, J. Math. Phys. 1998, 39:4343-4355
16. Xi-Guo Lee, Nuclear Physics Review, The Decomposable Theory of Gauge Potential and its Application, 2000, 17:201
17. Jia Duo-Jie, Duan Yi-Shi, Li Xi-Guo, Circulation Condition of the Two-Component Bose-Einstein Condensate, Phys. Lett. A, 2001, 289:245
18. Xi-Guo Lee, Duo-Jie Jia, Yuan Gao, Decomposition of the SU(2) Gauge Field and Circulation Condition of the Bose-Einstein Condensate, C.P.C., 2003, 27:12
19. Xi-Guo Lee, New Torsion Structure on Riemann-Cartan Manifold and Knots, C.P.C., 1999, 23:906
20. Xi-Guo Lee, Marcllo Baldo, Yi-Shi Duan, Torsion Structure in Riemann-Cartan Manifold and Dislocation, General Relativity and Gravity, 2002, 34(10):1569
21. Mo-Lin Ge, Rong-Gen Cai, Yu-Xiao Liu, *Memorial Volume for Yi-Shi Duan*, 2018, World Scientific