Traffic Flow Assignment Model and Algorithm Based on the Linear Inverse- Fundamental Diagram (Postprint)

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Abstract

In response to the problem that existing traffic flow assignment theory has difficulty handling road segment congestion states, a new static traffic flow assignment method under given road segment traffic conditions is proposed based on the linear inverse #1; fundamental diagram of traffic flow, density, and speed. First, by analyzing the fundamental diagram, a two-stage road segment travel time function is derived; second, by introducing a road segment traffic state indicator and constructing node flow conservation equations, system-optimal and user-equilibrium traffic flow assignment models are established; finally, for the non-convex user-equilibrium model, an effective branch-and-bound solution algorithm is designed by linearizing the non-convex terms in the objective function. Numerical examples verify the effectiveness of the new model and algorithm; the new theory extends the overly simplistic monotonically increasing characteristic assumption of existing road segment travel time functions, can effectively handle road segment congestion states, and improves the practicality of existing theories.

Full Text

Preamble

Traffic Assignment Model and Algorithm Based on Straight-Line Inverse Fundamental Diagram

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Abstract: Existing traffic assignment theory struggles to handle link congestion states effectively. To address this limitation, this paper proposes a novel static traffic assignment method that incorporates given link traffic states, grounded in the classic straight-line inverse fundamental diagram of traffic flow, density, and speed relationships. First, by analyzing this fundamental diagram, a two-stage link travel time function is derived. Second, by introducing link traffic state indicator variables and constructing node flow conservation equations, both system-optimal and user-equilibrium traffic assignment models are formulated. Finally, for the non-convex user-equilibrium model, an efficient branch-and-bound solution algorithm is designed through linearization of the non-convex terms in the objective function. Numerical examples validate the effectiveness of the proposed model and algorithm. The new theory extends the overly simplistic monotonic increasing characteristic assumption of existing link travel time functions, enabling effective handling of link congestion states and enhancing the practical utility of current theories.

Keywords: transportation planning; traffic assignment; congestion; non-convex programming

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0 Introduction

Static traffic assignment theory originated in 1952 when Wardrop proposed the two renowned principles of route choice: Wardrop' s first principle (user equilibrium) and Wardrop' s second principle (system optimization) [?]. In 1956, Beckman formulated the mathematical model corresponding to Wardrop' s first principle, known as the famous Beckman transformation [?]. Leblanc first introduced a practical and effective Frank-Wolfe algorithm for solving this model in 1975 [?]. Dafermos and Sparrow explored the use of variational inequalities for constructing traffic assignment models [?], while Daganzo and Sheffi pioneered the concept of stochastic traffic assignment [?]. In 2002, Bar-Gera proposed an origin-based algorithm that significantly improved the efficiency of solving static models [?]. Recent research has focused more on dynamic and quasi-dynamic traffic assignment theories, such as applying projection dynamic theory to model traffic assignment problems [?], and developing models based on variational inequality theory with side constraints to overcome the defect that link flows could exceed capacity in classical theory [?]. However, static traffic assignment theory generally assumes that link travel time functions exhibit monotonically increasing characteristics with respect to link flow, a hypothesis that contradicts the actual behavior of links in congested states. According to the fundamental diagram of traffic flow-density-speed relationships, when a link is congested, travel time decreases as link flow increases. Consequently, the travel time function for congested links should be monotonically decreasing with respect to flow. To resolve this issue, this paper employs the classic straight-line inverse fundamental diagram to derive a corresponding two-stage link travel time function, constructs new traffic assignment models using this two-stage travel time function, and designs effective solution algorithms.

The fundamental diagram of flow-density-speed relationships constitutes one of the foundational theories of traffic flow theory, widely applied across various domains of transportation research. For example, the classic Cell Transmission Model (CTM) is built upon this theory [?]. Compared with the common triangular flow-density-speed relationship, the inverse relationship depicted in Figure 1 [Figure 1: see original paper] captures the capacity drop phenomenon during the transition from free-flow to congested states—that is, the abrupt change from state C to state B in the diagram. As traffic density increases, the link first maintains a free-flow state; upon reaching the critical free-flow density q_{cr}, a sudden flow drop occurs from q_{max} to q_{max}^{jam}, after which the link enters a congested state. Clearly, the speed remains constant at v_f in free-flow conditions, while the speed at any congested state D corresponds to the slope λ of the line connecting the origin to point D.

The main innovations of this research include: (a) constructing traffic assignment models that consider two-stage link travel times based on the straight-line inverse fundamental diagram; and (b) designing an effective linear relaxation branch-and-bound solution algorithm for the non-convex user-equilibrium model.

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1 Link Travel Time Function

Figure 2 [Figure 2: see original paper] presents the two-stage link travel time function corresponding to the inverse flow-density-speed relationship. Assuming a link length of L, the following relationships clearly hold: $t_{free} = L/v_f$, so free-flow travel time is constant. Suppose the congested-state link travel time function takes the form $t = \alpha q + \beta$, where α and β are constants to be determined. Simple derivation shows that this congested-state travel time function corresponds to the linear relationship between flow and density during the congested phase in the fundamental diagram, with $\alpha < 0$ and $\beta > 0$. Let d_{jam} denote the jam density, then $\beta = \alpha d_{jam}$, and $q_{max} = \alpha(d_{jam} - d_{cr})$, while $q_{max}^{free} = q_{max}/v_f$.

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2 Traffic Flow Assignment Model

The variables and parameters related to the traffic assignment model with given link traffic states are defined as follows:

- a—denotes a directed link, $a \in A$ where A is the set of all directed links;
- n—denotes a node, $n \in N$ where N is the set of all nodes;
- A_n^+—the set of links entering node n, i.e., if $a \in A_n^+$, then n is the head node of directed link a;
- A_n^-—the set of links leaving node n, i.e., if $a \in A_n^-$, then n is the tail node of directed link a;

- b—denotes an OD pair, $b \in B$ where B is the set of all OD pairs;
- N_b^+—the origin node of OD pair b, with $N^+ \equiv \{n_b^+ | b \in B\}$;
- N_b^-—the destination node of OD pair b, with $N^- \equiv \{n_b^- | b \in B\}$;
- q_b—the travel demand for OD pair b;
- x_a—the flow on link a;
- x_a^b—the flow on link a for OD pair b;
- t_a^{free}—the travel time on link a in free-flow state, a given constant;
- t_a^{cr}—the travel time on link a in congested state when its flow is x_a;
- δ_a—the traffic state indicator for link a; $\delta_a = 1$ when link a is in free-flow state, otherwise $\delta_a = 0$ indicating congested state;
- q_a^{max}—the maximum service flow of link a in congested state;
- q_a^{free}—the maximum service flow of link a in free-flow state;
- Δ—a given minimum flow threshold in congested state.

The relevant constraints are:

Constraints (1)-(3) represent flow conservation equations at general nodes, OD origins, and OD destinations, respectively. Constraint (4) states that total link flow equals the sum of OD-specific link flows. Constraints (5) and (6) impose upper and lower bounds on link flows. Constraint (7) restricts the link traffic state indicator to be a binary variable.

From the fundamental relationship, we know $t_a^{cr} = \alpha_a x_a + \beta_a$ with $\alpha_a < 0$ and $\beta_a > 0$. The convex envelope of the concave function $\int_0^{x_a} t_a^{cr}(w)dw$ over $[0, q_a^{max}]$ is the linear function $\beta_a x_a$. Corresponding to traveler route choice behavior, two different objective functions can be formulated:

Equation (8) corresponds to system optimization (SO), while equation (9) corresponds to user equilibrium (UE). For convenience, the traffic assignment model with equation (8) as its objective is termed the two-stage system-optimal model, and the model with equation (9) as its objective is termed the two-stage user-equilibrium model.

Theorem 1. If for any link a, the free-flow travel time function is t_a^{free} and the congested-state travel time function takes the form $t_a^{cr} = \alpha_a x_a + \beta_a$, and the link traffic states δ_a are given, then the system-optimal traffic assignment model is a linear program.

Proof. When link traffic states δ_a are given, the constraint set becomes linear. The objective function component corresponding to free-flow states is linear in variables, and the component for congested states is also linear in variables. Combining these results proves the theorem.

Theorem 2. If for any link a, the free-flow travel time function is t_a^{free} and the congested-state travel time function takes the form $t_a^{cr} = \alpha_a x_a + \beta_a$, with $\alpha_a < 0$, $\beta_a > 0$, all δ_a given, and at least one link in set C has $\delta_a = 0$, then the user-equilibrium traffic assignment model is a non-convex program.

Proof. The objective function term corresponding to link $a \in C$ is $\int_0^{x_a}(\alpha_a w +$

$\beta_a)dw$, which is a concave function of variable x_a. The conclusion follows directly.

From Theorem 1, the system-optimal model is a linear program and can be solved efficiently. From Theorem 2, the user-equilibrium model is a non-convex program; the next section presents a corresponding linear relaxation branch-and-bound algorithm.

3 Linearization Relaxation Branch-and-Bound Algorithm

Let C denote the set of links in congested state, with $n = |C|$. From equation (9), the non-convex component in the user-equilibrium objective function is $\sum_{a\in C}\int_0^{x_a}(\alpha_a w + \beta_a)dw$. The variable x_a for congested links ranges over $[0, q_a^{max}]$. To simplify subsequent formulas, index the elements in set C and arrange the corresponding link flows in order to form vector $x_C = (x_1, x_2, ..., x_n)^T$, where the superscript T denotes transpose.

Let Z_{UE}^{lin} represent the linear function component of the user-equilibrium objective function after removing the non-convex terms. Define the set $S \equiv \{x_C \in \mathbb{R}^n | l \le x_C \le u\}$, where $l = (0, 0, ..., 0)^T$ and $u = (q_1^{max}, q_2^{max}, ..., q_n^{max})^T$.

The concave function $\sum_{j=1}^{n}\int_0^{x_j}(\alpha_j w + \beta_j)dw$ has a convex envelope that is a linear function over the rectangular region $[l, u]$. By replacing the non-convex user-equilibrium objective function with this linear relaxation, the original non-convex programming problem (NCP) is transformed into a linear programming problem (RLP), which provides a lower bound on the optimal value of the original problem. For links in free-flow state, the corresponding objective function component remains unchanged, making this a partial variable relaxation.

The branch-and-bound method described below solves a sequence of relaxed linear programs (RLP) to progressively improve the upper and lower bounds on the NCP optimal value until convergence. At iteration k of the algorithm, let L_k denote the collection of sub-rectangles that may contain the global optimal solution. For any $S \in L_k$, solving RLP(S) yields optimal solution x_S^k and optimal value $\mu(S)$. Then $\underline{\mu}_k = \min\{\mu(S) | S \in L_k\}$ is a lower bound on the NCP optimal value. If the optimal solution of RLP(S) is feasible for NCP, update the upper bound $\bar{\nu}_k$ on the NCP optimal value. Select a sub-rectangle $\hat{S} \in L_k$ such that $\mu(\hat{S}) = \underline{\mu}_k$, then bisect $\hat{S}$ along the direction perpendicular to its longest edge. Solve the corresponding RLP for each part and repeat this process until convergence. Note that x denotes the vector of OD-specific link flows, with $x \in S$ implicitly indicating $x_C \in S$.

Theorem 3. Assume the global optimal solution to the user-equilibrium non-convex program (NCP) exists. Then the algorithm either finds the global optimal solution in finite steps, or the limit point of the sequence $\{x^k\}$ generated by the algorithm must be a global optimal solution.

Proof. If the algorithm terminates in finite steps, the conclusion is obvious. When the algorithm generates an infinite iteration sequence, the exhaustive characteristic of the rectangular bisection method ensures that $\lim_{k\to\infty}(u_j^k - l_j^k) = 0$ for all j. From the convex envelope property of concave functions, the relationship between the original objective function and the relaxed linear programming objective function satisfies $\lim_{k\to\infty}[Z_{UE}(x^k) - Z_{UE}^{lin}(x^k)] = 0$. Combining these results yields $\lim_{k\to\infty} Z_{UE}(x^k) = Z_{UE}^*$, proving the proposition.

Although Theorem 3 proves that the algorithm can obtain the global optimal solution, note that the linear model after partial linear relaxation may have non-unique solutions, so the final global optimal solution may not be unique. For example, consider a network with only two paths connecting a single OD pair, where the paths share no common links. If the travel times on both paths are identical in free-flow state and the OD demand is small, this demand can be arbitrarily allocated between the two paths while both remain in free-flow state with unchanged travel times. In this case, we obtain different path flows (corresponding to different link flows) but identical objective function values. This example, where the flow assignment model is linear, explains why the global optimal solution may not be unique.

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4 Numerical Example Analysis

Since the system-optimal model is a linear program for which many efficient solution methods exist, this section focuses only on the non-convex user-equilibrium model. All time units are hours (hr) and distance units are kilometers (km). Flow units are pcu/hr and speed units are km/hr, where pcu represents passenger car units.

The algorithm from the previous section was implemented in Lingo, with the relaxed linear programming sub-models (RLP) solved directly using Lingo' s built-in solver. Computational efficiency is therefore closely related to the software' s own algorithms. For the example presented below, obtaining the final result required approximately 9 seconds, while solving the non-convex model directly with Lingo required about 4 seconds.

Figure 3 [Figure 3: see original paper] shows a traffic network with 10 nodes. Horizontal links have length 2 km, upper vertical links have length 1 km, lower vertical links have length 2 km, and diagonal link lengths are marked adjacent to the corresponding links in the figure. All nodes can serve as both origins and destinations for OD pairs, yielding 90 valid OD pairs. The corresponding OD travel demands are given in Table 1 , showing a total travel demand of 16,860 pcu/hr. All links are assumed to have a free-flow speed of 80 km/hr.

Table 2 provides link parameters for the network in Figure 3 [Figure 3: see original paper] and the computed link flows. Column (1) shows link flow values obtained using the branch-and-bound algorithm proposed in this paper; column

(2) shows link flow values obtained by solving the non-convex model directly with Lingo software; column (0) shows link flow values computed using existing traffic assignment models with side constraints under the assumption that all links are in free-flow state (i.e., ignoring link congestion). The column (0) results match those from the proposed model when all links are assumed to be in free-flow state. These results demonstrate that compared with existing side-constrained assignment models, the proposed model has broader applicability, capable of describing both free-flow states and the congested-state phenomenon where "link travel time decreases as flow increases."

Table 3 shows how the upper and lower bounds of the objective function change during the solution process for the column (1) link flows. The algorithm obtains the global optimal solution after 10 iterations (21 RLP sub-model solves), with an objective function value of 16,464.88. Directly solving the non-convex model with Lingo yields an objective function value of 16,585.31. However, note that the computed link flow values differ between the two methods, as shown in columns (1) and (2) of Table 2. Table 4 presents the objective values of the 21 RLP sub-models (OLP) and the corresponding original user-equilibrium model objective values (OUE). These computational results demonstrate that the algorithm proposed in this paper outperforms Lingo' s built-in non-convex model solver.

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5 Conclusion

By analyzing the straight-line inverse fundamental diagram, this paper establishes a static traffic assignment model with two-stage link travel times. For the non-convex user-equilibrium model, a partial linearization relaxation branch-and-bound solution algorithm is designed. Numerical analysis validates the effectiveness of both the model and algorithm. Directions for future research include: (a) considering more complex traffic flow-density-speed fundamental diagrams, such as curved inverse relationships; (b) investigating quasi-dynamic traffic assignment with time-dependent link state updates; and (c) extending to multi-modal joint assignment models.

References

[1] Wardrop J G. Some theoretical aspects of road traffic research [J]. Proceedings of the Institution of Civil Engineers Part II, 1952, 1 (5): 325–362.

[2] Sheffi Y. Urban Transportation networks: equilibrium analysis with mathematical programming methods [M]. Englewood Cliffs: Prentice-Hall, Inc., 1985.

[3] Dafermos S C, Sparrow F T. The traffic assignment problem for a general network [J]. J. Res. Natl. Bureau Stand., Ser. B, 1969, 73 (2): 91–118.

[4] Daganzo C F, Sheffi Y. On stochastic models of traffic assignment [J]. Transportation Science, 1977, 11 (3): 253–274.

[5] Bar-Gera H. Origin-based algorithm for the traffic assignment problem [J]. Transportation Science, 2002, 36 (4): 398–417.

[6] He Shengxue, Fan Bingquan. Multi-user dynamic traffic flow assignment model and algorithm research [J]. Journal of University of Shanghai for Science and Technology, 2006, 28 (5): 460-464.

[7] He Shengxue, Dong Qiong, Xu Fuyuan. Traffic flow assignment model based on network dual equilibrium [J]. Highway Traffic Science and Technology, 2010, 27 (9): 105-110.

[8] He Shengxue, He Jianjia, Xu Fuyuan. Traffic flow assignment model with side constraints based on network dual equilibrium [J]. Journal of Transportation Systems Engineering and Information Technology, 2011, 11 (2): 100-105.

[9] He Shengxue, Li Qiuman. Network traffic flow assignment with side constraints and its application [J]. Computer Applications Research, 2011, 28 (7): 2595-2598.

[10] Xie J, Xie C. New insights and improvements of using paired alternative segments for traffic assignment [J]. Transportation Research Part B, 2016, 93 (4): 406–424.

[11] Ma W, Qian S. On the variance of recurrent traffic flow for statistical traffic assignment [J]. Transportation Research Part C, 2017, 81 (1): 57–82.

[12] Guo R Y, Huang H J. A discrete dynamical system of formulating traffic assignment: Revisiting Smith' s model [J]. Transportation Research Part C, 2016, 71 (2): 122–142.

[13] Daganzo C F. The cell transmission model: a dynamic representation of highway traffic consistent with the hydrodynamic theory. Transportation Research Part B, 1994, 28 (4), 269–287.

[14] Daganzo C F. The cell transmission model, part II: network traffic [J]. Transportation Research Part B, 1995, 29 (2), 79–93.

[15] He S X. Will a higher free-flow speed lead us to a less congested freeway? [J]. Transportation Research Part A, 2016, 85 (1): 17-38.

Note: Figure translations are in progress. See original paper for figures.