# DOA Sparse Reconstruction Method Based on Low-Rank Matrix Recovery (Postprint)

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## Abstract

To improve the estimation accuracy and resolution of direction-of-arrival (DOA) angle estimation algorithms under non-uniform noise, a weighted L1 sparse reconstruction DOA estimation algorithm in the second-order statistical domain is proposed based on low-rank matrix recovery theory. The algorithm employs a low-rank matrix recovery approach, introducing an elastic regularization factor to transform the received signal covariance matrix reconstruction problem into a semidefinite programming (SDP) problem that can be efficiently solved, thereby reconstructing the noise-free covariance matrix; subsequently, DOA parameter estimation is achieved by utilizing sparse reconstruction weighted L1 norm in the second-order statistical domain. Numerical simulations demonstrate that, compared with traditional MUSIC, L1-SVD, and weighted L1 algorithms, the proposed algorithm can significantly suppress the influence of non-uniform noise and exhibits superior DOA parameter estimation performance, and under low signal-to-noise ratio conditions, the proposed algorithm possesses higher angular resolution and estimation accuracy.

## Full Text

## Preamble

### Low-rank Matrix Recovery Based DOA Sparse Reconstruction Method

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**Abstract:** To improve the estimation accuracy and resolution of Direction-of-Arrival (DOA) algorithms under non-uniform noise, this paper proposes a weighted L1 sparse reconstruction DOA estimation algorithm in the second-order statistical domain based on low-rank matrix recovery theory. The algorithm transforms the received signal covariance matrix reconstruction problem into a semidefinite programming (SDP) problem that can be efficiently solved by introducing an elastic regularization factor, thereby reconstructing the noise-free covariance matrix. DOA parameters are then estimated using sparse reconstruction with weighted L1 norm in the second-order statistical domain. Numerical simulations demonstrate that compared with traditional MUSIC, L1-SVD, and weighted L1 algorithms, the proposed algorithm can significantly suppress the influence of non-uniform noise and achieve superior DOA parameter estimation performance, with higher angular resolution and estimation accuracy under low SNR conditions.

**Keywords:** direction of arrival (DOA); nonuniform noise; low-rank matrix recovery; second-order statistics; weighted L1 norm

## 0 Introduction

Direction-of-Arrival (DOA) estimation, also known as direction finding (DF), represents a fundamental problem in array signal processing with applications in radar, mobile communications, wireless sensor networks, and radio astronomy. Traditional subspace-based DOA estimation algorithms, such as the Multiple Signal Classification (MUSIC) algorithm, significantly improve DOA estimation accuracy and resolution by exploiting signal and noise subspaces. However, these subspace methods typically assume zero-mean complex Gaussian white noise with unit variance. When the additive noise at array sensors is non-uniform Gaussian noise, eigen-decomposition of the received signal covariance matrix causes signal subspace leakage, severely degrading or even invalidating the performance of subspace-based algorithms.

To address this limitation, researchers have proposed various DOA estimation algorithms for non-uniform Gaussian noise environments. Reference [5] presents a maximum likelihood (ML) estimation algorithm that iteratively solves the log-likelihood function for signals and noise. However, the ML algorithm' s strong dependence on initial values and high computational complexity limit its practical application. Reference [6] introduces a matrix completion-based MUSIC (MC-MUSIC) algorithm that reconstructs the noise-free signal covariance using matrix completion principles before applying traditional MUSIC for DOA estimation. This approach reduces non-uniform noise effects while avoiding iterative solutions. Nevertheless, MC-MUSIC fails to consider correlations among covariance matrix elements, potentially leading to numerical instability and poor algorithm robustness.

More recently, leveraging the spatial sparsity of target signals, researchers have applied compressive sensing theory to DOA estimation. Liang G et al. proposed a sparse reconstruction algorithm that achieves high-precision DOA estimates when the number of sources is known. However, under non-uniform Gaussian noise or unknown source prior information, this algorithm cannot resolve closely spaced angles, resulting in poor spatial resolution.

To overcome these challenges, this paper proposes a weighted L1 sparse reconstruction DOA estimation algorithm in the second-order statistical domain based on low-rank matrix recovery theory (LR-WLOSRSS). Under non-uniform Gaussian noise, the algorithm first introduces an elastic regularization factor to transform the received signal covariance matrix reconstruction problem into an efficiently solvable SDP problem, reconstructing the noise-free covariance matrix and eliminating non-uniform noise effects while improving numerical stability. Subsequently, weighted L1 norm (WL1) is employed in the second-order statistical domain for DOA parameter estimation. Numerical simulations show that compared with traditional MUSIC, L1-SVD, and weighted L1 algorithms, the proposed algorithm significantly suppresses non-uniform noise, achieves superior DOA estimation performance, and provides higher angular resolution and better estimation accuracy under low SNR conditions.

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## 2 Noise-Free Signal Covariance Reconstruction via Low-Rank Matrix Recovery

Consider $Q$ far-field narrowband signals impinging on a uniform linear array with $M$ elements. The array received signal model can be expressed as:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)$$

where $\mathbf{x}(t)$ is the received signal vector, $\mathbf{s}(t)$ and $\mathbf{n}(t)$ are the signal amplitude and non-uniform Gaussian noise vectors, respectively, and $\mathbf{A}$ is the array steering matrix. Assuming the number of array elements far exceeds the number of signals ($M \gg Q$), the noise-free signal covariance matrix $\mathbf{R}_0 = \mathbf{A}\mathbf{P}\mathbf{A}^H$ has rank $Q$, making it a low-rank matrix. The off-diagonal elements of the covariance matrix $\mathbf{R}$ are equivalent to the corresponding elements of the received signal covariance, enabling reconstruction of the diagonal elements through low-rank matrix recovery theory to eliminate non-uniform Gaussian noise effects.

For a given matrix $\mathbf{X} \in \mathbb{C}^{m\times n}$, its projection onto subset $\Omega$ (sampling operation) is defined as:

$$[\mathcal{P}_\Omega(\mathbf{X})]_{ij} = \begin{cases} X_{ij} & (i,j) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

where $\Omega \subseteq [m] \times [n]$ represents the index set of sampled elements. The low-rank matrix recovery model can be formulated as:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \operatorname{rank}(\mathbf{X}) \\ \text{s.t.} \quad & \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{X}_0) \end{aligned}$$

Due to the non-convex nature of the rank function, we relax it to the convex nuclear norm:

$$\|\mathbf{X}\|_* = \sum_i \sigma_i(\mathbf{X})$$

where $\sigma_i(\mathbf{X})$ are the singular values of $\mathbf{X}$. To improve numerical stability when solving strongly correlated data, we introduce an elastic regularization term $\frac{\tau}{2}\|\mathbf{X}\|_F^2$, yielding:

$$\min_{\mathbf{X}} \|\mathbf{X}\|_* + \frac{\tau}{2}\|\mathbf{X}\|_F^2 \quad \text{s.t.} \quad \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{X}_0)$$

where $\tau$ balances the nuclear norm and Frobenius norm regularization.

Based on the signal covariance model $\mathbf{R} = \mathbf{R}_0 + \mathbf{W}$, where $\mathbf{W}$ is the diagonal noise covariance matrix, the low-rank recovery formulation becomes:

$$\begin{aligned} \min_{\mathbf{R}_0} \quad & \|\mathbf{R}_0\|_* + \frac{\tau}{2}\|\mathbf{R}_0\|_F^2 \\ \text{s.t.} \quad & \mathcal{P}_\Omega(\mathbf{R}_0) = \mathcal{P}_\Omega(\mathbf{R}) \end{aligned}$$

Since $\mathbf{R}_0$ is positive semidefinite, we have $\|\mathbf{R}_0\|_* = \operatorname{tr}(\mathbf{R}_0)$. The constraint can be expressed using a selection matrix $\mathbf{J}$ that extracts the diagonal elements:

$$\mathbf{J}\operatorname{vec}(\mathbf{R}_0) = \mathbf{0}$$

where $\mathbf{J} \in \mathbb{R}^{M\times M^2}$ is a selection matrix. This leads to the convex optimization problem:

$$\begin{aligned} \min_{\mathbf{R}_0} \quad & \operatorname{tr}(\mathbf{R}_0) + \frac{\tau}{2}\|\mathbf{R}_0\|_F^2 \\ \text{s.t.} \quad & \|\mathbf{J}\operatorname{vec}(\mathbf{R}_0) - \mathbf{J}\operatorname{vec}(\mathbf{R})\|_2 \le \xi \end{aligned}$$

where $\xi$ is an error tolerance parameter. Using the Schur complement theorem, this can be transformed into an SDP problem solvable efficiently via CVX toolbox. The reconstructed noise-free covariance $\mathbf{R}_0$ can then be used with traditional subspace methods for DOA estimation.

## 3 Weighted L1 Sparse Reconstruction in Second-Order Statistical Domain

To further improve estimation accuracy and resolution under low SNR while reducing computational complexity, we propose a weighted L1 norm sparse reconstruction algorithm in the second-order statistical domain. The elements of the noise-free covariance matrix $\mathbf{R}_0$ represent correlation coefficients between array outputs:

$$R_0(i,j) = \sum_{q=1}^{Q} P_q e^{j\pi(i-j)\sin\theta_q}$$

where $P_q$ is the power of the $q$-th signal and $\theta_q$ is its DOA. Expanding $\mathbf{R}_0$ row-wise yields a vector $\mathbf{r} \in \mathbb{C}^{M^2}$ containing redundant information, as off-diagonal elements share the same coefficient characteristics. This allows conversion of the multi-vector problem into a single-vector problem through averaging in the second-order statistical domain.

The averaged vector $\mathbf{r}_a$ is obtained by summing along anti-diagonals:

$$r_a(m) = \frac{1}{M-|m|} \sum_{n=1}^{M-|m|} R_0(n, n+m), \quad m = -(M-1), \dots, M-1$$

This can be expressed as $\mathbf{r}_a = \mathcal{B}\mathbf{p}$, where $\mathcal{B}$ is a virtual array manifold matrix and $\mathbf{p}$ is the signal power vector. Discretizing the spatial domain into $N$ grids $\{\theta_1, \theta_2, \dots, \theta_N\}$, we obtain the sparse representation:

$$\mathbf{r}_a = \mathbf{B}(\theta)\mathbf{P} + \epsilon$$

where $\mathbf{B}(\theta) \in \mathbb{C}^{(2M-1)\times N}$ is the overcomplete basis matrix and $\mathbf{P}$ is the sparse power vector.

The sparse reconstruction problem is formulated as:

$$\min_{\mathbf{P}} \|\mathbf{P}\|_1 \quad \text{s.t.} \quad \|\mathbf{r}_a - \mathbf{B}\mathbf{P}\|_2 \le \eta$$

However, the standard L1 norm penalizes large coefficients more heavily than small ones, resulting in biased estimates. To address this, we employ weighted L1 minimization:

$$\min_{\mathbf{P}} \sum_{i=1}^{N} w_i |P_i| \quad \text{s.t.} \quad \|\mathbf{r}_a - \mathbf{B}\mathbf{P}\|_2 \le \eta$$

where the weights are updated iteratively as $w_i^{(m)} = 1/(|P_i^{(m-1)}| + \epsilon)$ to enhance sparsity. This convex optimization problem can be solved using second-order cone programming (SOCP).

The complete LR-WLOSRSS algorithm is summarized as:

1. **Input**: Received signal covariance $\mathbf{R}$, array manifold $\mathbf{A}$, grid points $\theta$, regularization parameters $\tau, \eta, \epsilon$, maximum iterations $MaxIter$
2. **Step 1**: Solve the SDP problem to reconstruct noise-free covariance $\mathbf{R}_0$
3. **Step 2**: Compute averaged vector $\mathbf{r}_a = \text{avr}(\mathbf{R}_0)$
4. **Step 3**: Initialize weights $\mathbf{w}^{(0)} = \mathbf{1}$ and iteration counter $m = 0$
5. **Step 4**: Solve weighted L1 problem to obtain $\mathbf{P}^{(m)}$
6. **Step 5**: Update weights $w_i^{(m+1)} = 1/(|P_i^{(m)}| + \epsilon)$
7. **Step 6**: Repeat until $\|\mathbf{P}^{(m)} - \mathbf{P}^{(m-1)}\|_2 \leq \xi$ or $m \geq MaxIter$
8. **Output**: DOA estimates from peaks of $\mathbf{P}$

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## 4 Experimental Simulations and Analysis

This section validates the proposed algorithm through numerical simulations. Simulation parameters: uniform linear array with $M = 8$ elements, $L = 500$ snapshots, signal power $\sigma_s^2 = 5$, error constant $\xi = 0.01$, grid number $N = 181$, and SNR defined as $10\log_{10}(\sigma_s^2/\sigma_n^2)$. Non-uniform Gaussian noise covariance is $\mathbf{W} = \text{diag}\{2.0, 10, 2.5, 5.0, 0.5, 1.5, 3.0, 5.0\}$. RMSE is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{KQ}\sum_{k=1}^{K}\sum_{q=1}^{Q}(\hat{\theta}_{k,q} - \theta_q)^2}$$

where $K$ is the number of Monte Carlo trials.

**Experiment 1**: Non-coherent signals at $-3°$, $10°$, and $16°$ with SNR = 0 dB and $-5$ dB. [Figure 1: see original paper] shows the spatial spectra. At 0 dB, MUSIC, WL1, and L1-SVD cannot resolve the $10°$ and $16°$ targets, while MC-MUSIC and LR-WLOSRSS succeed. At $-5$ dB, only LR-WLOSRSS effectively resolves all three angles, demonstrating superior performance in non-uniform noise and low SNR conditions with narrower mainlobes and lower sidelobes.

**Experiment 2**: Signals at $-3°$, $10°$, and $13°$ with SNR = 5 dB. [Figure 2: see original paper] compares spatial spectra. MUSIC and WL1 fail to resolve $10°$ and $13°$. MC-MUSIC, though reconstructing noise-free covariance, lacks sufficient resolution. LR-WLOSRSS and ideal L1-SVD resolve the closely spaced angles, with LR-WLOSRSS showing higher accuracy and lower sidelobes.

**Experiment 3**: Three signals at $-3°$, $10°$, and $16°$ with SNR = 5 dB, varying source number $K$. [Figure 3: see original paper] shows that traditional MUSIC requires exact prior knowledge of source number ($K = 3$) to resolve all angles, while LR-WLOSRSS remains effective for $K = 1, 2, 3$, demonstrating robustness to source number mismatch.

**Experiment 4**: Signals at $-3^\circ$, $10^\circ$, and $13^\circ$ with $L = 500$ snapshots, SNR ranging from $-8$ dB to 12 dB, and 200 Monte Carlo trials. [Figure 4: see original paper] and show RMSE versus SNR. Traditional MUSIC exhibits high RMSE at low SNR. MC-MUSIC outperforms MUSIC by eliminating non-uniform noise via matrix completion. WL1 and L1-SVD show intermediate performance. LR-WLOSRSS achieves the lowest RMSE across all SNRs, particularly excelling in low-SNR regimes.

**Experiment 5**: Two signals at $-3^\circ$ and $5^\circ$ with SNR $= 0$ dB, varying snapshots $L$ from 100 to 1200. [Figure 5: see original paper] and demonstrate that RMSE decreases with increasing snapshots for all algorithms. LR-WLOSRSS consistently maintains lower RMSE than MUSIC, MC-MUSIC, WL1, and L1-SVD, confirming superior performance under non-uniform Gaussian noise.

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## 5 Conclusion

To improve DOA estimation accuracy and resolution under non-uniform noise, this paper proposes a weighted L1 sparse reconstruction algorithm (LR-WLOSRSS) in the second-order statistical domain based on low-rank matrix recovery. The algorithm transforms the covariance matrix reconstruction problem into an efficiently solvable SDP problem using elastic regularization, then converts the multi-vector sparse reconstruction into a single-vector problem via averaging in the second-order statistical domain, and finally employs weighted L1 norm for sparse reconstruction. Simulation results demonstrate that compared with MUSIC, MC-MUSIC, WL1, and L1-SVD, the proposed algorithm achieves superior DOA estimation performance under non-uniform Gaussian noise and low SNR conditions, with higher resolution and accuracy.

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*Note: Figure translations are in progress. See original paper for figures.*