Postprint of Multi-Attribute Group Decision-Making Method Based on Pythagorean Fuzzy Linguistic Power Weighted Averaging Operator

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Abstract

To address multi-attribute group decision-making problems, this paper employs Pythagorean fuzzy linguistic sets, which facilitate experts in utilizing linguistic term set information for evaluation and provide flexible value assignment. First, based on distance measures of linguistic term sets and Pythagorean fuzzy sets, the definition and related properties of the distance measure for Pythagorean fuzzy linguistic numbers are presented. Subsequently, utilizing the distance measure of Pythagorean fuzzy linguistic numbers as the distance metric for the power average (PA) operator, the Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator is proposed to aggregate different expert evaluation matrices in the group decision-making process while simultaneously accounting for differences among expert evaluations during aggregation. Finally, a novel group decision-making method in the Pythagorean fuzzy linguistic environment is constructed based on the PFLPWA operator, and the effectiveness and applicability of the PFLPWA operator in group decision-making are verified through a case study.

Full Text

Preamble

Title: Multi-Attribute Group Decision-Making Method Based on Pythagorean Fuzzy Linguistic Power Weighted Average Operator

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Abstract: For multi-attribute group decision-making (MAGDM) problems, this study employs Pythagorean fuzzy linguistic numbers as input information, which enables decision-makers to evaluate alternatives effectively with more flexible values. First, a Pythagorean fuzzy linguistic distance measure is created based on distance measures for Pythagorean fuzzy sets and linguistic sets, and its properties are provided. Then, by using this distance measure in the power average operator, the Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator is proposed to aggregate a collection of Pythagorean fuzzy linguistic numbers (PFLNs), where the differences among decision-makers are considered during the aggregation process. Finally, a group decision-making method is constructed based on the PFLPWA operator, and a numerical example is given to demonstrate its effectiveness and feasibility.

Keywords: distance measure; Pythagorean fuzzy linguistic number; Pythagorean fuzzy linguistic power weighted average operator; multi-attribute group decision-making

0 Introduction

Both intuitionistic fuzzy sets and Pythagorean fuzzy sets express membership and non-membership degrees as specific real numbers in the unit interval. However, in multi-attribute decision-making problems, due to the complexity of alternatives, evaluators may not always be capable of providing precise membership and non-membership values. In such situations, intuitionistic fuzzy sets and Pythagorean fuzzy sets are not suitable as effective tools for evaluating alternatives.

Pythagorean fuzzy sets, extended from intuitionistic fuzzy sets, describe object attributes through membership and non-membership degrees that must satisfy the constraint that the sum of their squares is less than or equal to 1. Geometrically, the constraint region for membership and non-membership degrees expands from a triangular region to a quarter-circle region, indicating that any intuitionistic fuzzy set’ s values can be accommodated within the Pythagorean fuzzy set’ s value region. Therefore, compared to intuitionistic fuzzy sets, Pythagorean fuzzy sets provide evaluators with more flexible evaluation scales. Recent studies have demonstrated the superiority of Pythagorean fuzzy sets in handling multi-attribute decision-making problems.

By analyzing the expression forms of membership and non-membership values, we observe that both intuitionistic fuzzy sets and Pythagorean fuzzy sets use specific real numbers in the unit interval. However, in multi-attribute decision-making problems, due to alternative complexity, evaluators may lack the ability to provide precise membership and non-membership values. Recognizing this limitation, Wang Jianqiang and Li Jingjing extended linguistic sets to intuitionistic fuzzy sets and proposed intuitionistic fuzzy linguistic sets. Evaluators can use a linguistic term from a linguistic term set to describe an alternative' s performance under an attribute, employing membership and non-membership degrees to characterize the extent of belonging and non-belonging to the linguistic term, thereby yielding intuitionistic fuzzy linguistic numbers. Nevertheless, literature indicates that when evaluators provide assessment values for alternatives regarding certain attributes, they cannot offer situations where the sum of belonging and non-belonging degrees to linguistic terms exceeds 1. Therefore, Peng Xindong and Yang Yong extended linguistic sets to Pythagorean fuzzy sets, proposed Pythagorean fuzzy linguistic sets, defined their basic operational rules, and applied them to multi-attribute group decision-making.

Given that Pythagorean fuzzy linguistic sets combine the flexible evaluation scale of Pythagorean fuzzy sets with the convenience of linguistic sets for decision-maker expression, this paper further considers their application to multi-attribute group decision-making problems. In such problems, different experts provide individual evaluation matrices, and effectively fusing these into a comprehensive matrix represents a key step. To this end, this paper extends the power average (PA) operator to Pythagorean fuzzy linguistic sets and proposes a Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator based on Pythagorean fuzzy linguistic distance measures, which simultaneously considers both objective weights and subjective weights. The paper also examines fundamental properties such as idempotency, commutativity, and monotonicity, as well as special cases. Subsequently, a group decision-making method is constructed using the PFLPWA operator with detailed steps provided. Finally, case analysis demonstrates the method' s feasibility.

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1 Preliminaries

1.1 Intuitionistic Fuzzy Linguistic Set

Definition 1. Let $S = \{s_\theta | \theta = 0, 1, \dots, g\}$ where g is a positive integer representing possible values of a linguistic variable, satisfying: (a) if $\theta > \delta$, then $s_\theta > s_\delta$; (b) there exists a negation operator $\mathrm{neg}(s_\theta) = s_{g-\theta}$. Then S is called a discrete linguistic term set. For example, when $g = 8$, the linguistic term set is defined as:

$$S = \{s_0, s_1, \dots, s_8\} = \{\text{极差, 很差, 差, 略差, 一般, 略好, 好, 很好, 极好}\}$$

For convenience in computation between specific linguistic terms, the discrete linguistic term set is extended to a continuous linguistic term set $\bar{S} = \{s_\theta | \theta \in [0, g]\}$, which also satisfies the above conditions.

Definition 2. Let $X = \{x_1, x_2, \dots, x_n\}$ be a given set. An intuitionistic fuzzy set I on X is defined as:

$$I = \{\langle x_i, \mu_I(x_i), v_I(x_i)\rangle | x_i \in X\}$$

where $\mu_I(x_i)$ and $v_I(x_i)$ represent the membership and non-membership degrees of element x_i to set I, respectively. The pair $\langle \mu_I(x_i), v_I(x_i)\rangle$ is called an intuitionistic fuzzy number (IFN), conveniently denoted as $\langle \mu, v\rangle$.

Definition 3. Let $X = \{x_1, x_2, \dots, x_n\}$ be a given set. An intuitionistic fuzzy linguistic set A on X is defined as:

$$A = \{\langle x_i, s_{\theta(x_i)}, \mu_A(x_i), v_A(x_i)\rangle | x_i \in X\}$$

where $s_{\theta(x_i)} \in S$ is the linguistic evaluation value provided by decision-makers for element x_i, and $\mu_A(x_i)$ and $v_A(x_i)$ represent the membership and non-membership degrees of x_i belonging to linguistic value $s_{\theta(x_i)}$, satisfying $\mu_A(x_i) + v_A(x_i) \leq 1$ and $\mu_A(x_i), v_A(x_i) \in [0, 1]$. The triple $\langle s_{\theta(x_i)}, \mu_A(x_i), v_A(x_i)\rangle$ is called an intuitionistic fuzzy linguistic number (IFLN), conveniently denoted as $\langle s_\theta, \mu, v\rangle$.

1.2 Pythagorean Fuzzy Linguistic Set

Definition 4. Let $X = \{x_1, x_2, \dots, x_n\}$ be a given set. A Pythagorean fuzzy set P on X is defined as:

$$P = \{\langle x_i, \mu_P(x_i), v_P(x_i)\rangle | x_i \in X\}$$

where $\mu_P(x_i)$ and $v_P(x_i)$ represent the membership and non-membership degrees of element x_i to set P, respectively. The pair $\langle \mu_P(x_i), v_P(x_i)\rangle$ is called a Pythagorean fuzzy number (PFN), conveniently denoted as $\langle \mu, v\rangle$, with $\mu, v \in [0, 1]$ and $\mu^2 + v^2 \leq 1$.

Definition 5. Let $X = \{x_1, x_2, \dots, x_n\}$ be a given set. A Pythagorean fuzzy linguistic set A on X is defined as:

$$A = \{\langle x_i, s_{\theta(x_i)}, \mu_A(x_i), v_A(x_i)\rangle | x_i \in X\}$$

where $s_{\theta(x_i)} \in S$ is the linguistic evaluation value, and $\mu_A(x_i)$ and $v_A(x_i)$ represent the degrees of belonging and not belonging to linguistic value $s_{\theta(x_i)}$, satisfying $\mu_A(x_i)^2 + v_A(x_i)^2 \leq 1$ and $\mu_A(x_i), v_A(x_i) \in [0, 1]$. The triple $\langle s_{\theta(x_i)}, \mu_A(x_i), v_A(x_i)\rangle$ is called a Pythagorean fuzzy linguistic number (PFLN), conveniently denoted as $\langle s_\theta, \mu, v\rangle$.

Definition 6. Let $\alpha = \langle s_{\theta(\alpha)}, \mu_\alpha, v_\alpha\rangle$ and $\beta = \langle s_{\theta(\beta)}, \mu_\beta, v_\beta\rangle$ be two Pythagorean fuzzy linguistic numbers. The natural partial order is defined as follows: 1. If $s_{\theta(\alpha)} < s_{\theta(\beta)}$, then $\alpha < \beta$; 2. If $s_{\theta(\alpha)} = s_{\theta(\beta)}$, then: - If $\mu_\alpha < \mu_\beta$ and $v_\alpha \geq v_\beta$, then $\alpha < \beta$; - If $\mu_\alpha = \mu_\beta$ and $v_\alpha = v_\beta$, then $\alpha = \beta$.

Definition 7. Let $\alpha = \langle s_{\theta(\alpha)}, \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle s_{\theta(\beta)}, \mu_\beta, v_\beta \rangle$ be two Pythagorean fuzzy linguistic numbers. Their basic operations are defined as: 1. $\alpha \oplus \beta = \langle s_{\theta(\alpha)+\theta(\beta)}, \sqrt{\mu_\alpha^2 + \mu_\beta^2 - \mu_\alpha^2 \mu_\beta^2}, v_\alpha v_\beta \rangle$; 2. $\alpha \otimes \beta = \langle s_{\theta(\alpha)\times\theta(\beta)}, \mu_\alpha \mu_\beta, \sqrt{v_\alpha^2 + v_\beta^2 - v_\alpha^2 v_\beta^2} \rangle$; 3. $\lambda \cdot \alpha = \langle s_{\lambda\theta(\alpha)}, \sqrt{1-(1-\mu_\alpha^2)^\lambda}, v_\alpha^\lambda \rangle$, $\lambda \geq 0$; 4. $\alpha^\lambda = \langle s_{\theta(\alpha)^\lambda}, \mu_\alpha^\lambda, \sqrt{1-(1-v_\alpha^2)^\lambda} \rangle$, $\lambda \geq 0$.

Definition 8. The Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator is defined as a mapping $P : \mathbb{R}^n \to \mathbb{R}$ that satisfies:

$$\mathrm{PA}(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^{n} (1 + T(a_i)) a_i}{\sum_{i=1}^{n} (1 + T(a_i))}$$

where $T(a_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Supp}(a_i, a_j)$, and $\mathrm{Supp}(a_i, a_j)$ denotes the support degree for a_i from a_j, satisfying: 1. $\mathrm{Supp}(a_i, a_j) \in [0, 1]$; 2. $\mathrm{Supp}(a_i, a_j) = \mathrm{Supp}(a_j, a_i)$; 3. If $|a_i - a_j| < |a_s - a_t|$, then $\mathrm{Supp}(a_i, a_j) \geq \mathrm{Supp}(a_s, a_t)$.

The weights in the power average operator are primarily composed of the mutual support degrees among objective arrays, which can objectively reflect the differences between each element and other elements in the array.

1.3 Power Average Operator

Literature [12] proposed a power average operator that considers the difference degree between an individual information set and the overall set. The relevant concepts are presented below.

Definition 9. The power average operator is a mapping $PA : \mathbb{R}^n \to \mathbb{R}$ defined as:

$$\mathrm{PA}(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^{n} (1 + T(a_i)) a_i}{\sum_{i=1}^{n} (1 + T(a_i))}$$

where $T(a_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Supp}(a_i, a_j)$ represents the support degree for a_i from all other arguments, and $\mathrm{Supp}(a_i, a_j)$ is the support degree for a_i from a_j, satisfying: 1. $\mathrm{Supp}(a_i, a_j) \in [0, 1]$; 2. $\mathrm{Supp}(a_i, a_j) = \mathrm{Supp}(a_j, a_i)$; 3. If $|a_i - a_j| < |a_s - a_t|$, then $\mathrm{Supp}(a_i, a_j) \geq \mathrm{Supp}(a_s, a_t)$.

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2 Pythagorean Fuzzy Linguistic Power Weighted Average Operator

2.1 Distance Measure for Pythagorean Fuzzy Linguistic Numbers

By combining distance measures for linguistic terms and Pythagorean fuzzy numbers, we define the distance measure for Pythagorean fuzzy linguistic numbers and investigate its fundamental properties.

Definition 10. Let $\alpha = \langle s_{\theta(\alpha)}, \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle s_{\theta(\beta)}, \mu_\beta, v_\beta \rangle$ be two Pythagorean fuzzy linguistic numbers. The distance measure is defined as:

$$d(\alpha, \beta) = \frac{1}{4}\left(\left|\frac{2(\theta(\alpha) - \theta(\beta))}{g}\right| + |\mu_\alpha^2 - \mu_\beta^2| + |v_\alpha^2 - v_\beta^2| + |\pi_\alpha^2 - \pi_\beta^2|\right)$$

where $\pi_\alpha = \sqrt{1 - \mu_\alpha^2 - v_\alpha^2}$ and $\pi_\beta = \sqrt{1 - \mu_\beta^2 - v_\beta^2}$.

Property 1. Let $\alpha = \langle s_{\theta(\alpha)}, \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle s_{\theta(\beta)}, \mu_\beta, v_\beta \rangle$ be two Pythagorean fuzzy linguistic numbers. The distance measure $d(\alpha, \beta)$ satisfies the following properties: 1. $0 \leq d(\alpha, \beta) \leq 1$; 2. $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$; 3. $d(\alpha, \beta) = d(\beta, \alpha)$; 4. If $\alpha \leq \beta \leq \gamma$, then $d(\alpha, \beta) \leq d(\alpha, \gamma)$ and $d(\beta, \gamma) \leq d(\alpha, \gamma)$.

Proof. 1. According to Definition 10, we have:

$$0 \leq \left|\frac{2(\theta(\alpha) - \theta(\beta))}{g}\right| \leq 2, \quad 0 \leq |\mu_\alpha^2 - \mu_\beta^2| \leq 1, \quad 0 \leq |v_\alpha^2 - v_\beta^2| \leq 1, \quad 0 \leq |\pi_\alpha^2 - \pi_\beta^2| \leq 1$$

Therefore, $0 \leq d(\alpha, \beta) \leq 1$.

2. If $\alpha = \beta$, then $\theta(\alpha) = \theta(\beta)$, $\mu_\alpha = \mu_\beta$, $v_\alpha = v_\beta$, and $\pi_\alpha = \pi_\beta$, which yields $d(\alpha, \beta) = 0$. Conversely, if $d(\alpha, \beta) = 0$, then all absolute difference terms must be zero, implying $\alpha = \beta$.

3. Symmetry follows directly from the absolute value operations in Definition 10.

4. For monotonicity, when $\alpha \leq \beta \leq \gamma$, we have $\theta(\alpha) \leq \theta(\beta) \leq \theta(\gamma)$, $\mu_\alpha \leq \mu_\beta \leq \mu_\gamma$, and $v_\alpha \geq v_\beta \geq v_\gamma$. Consequently, the differences between α and γ dominate those between α and β or β and γ, establishing the inequality.

2.2 Pythagorean Fuzzy Linguistic Power Weighted Average (PFLPWA) Operator

To effectively aggregate Pythagorean fuzzy linguistic arrays, we extend the power average operator to Pythagorean fuzzy sets based on the proposed distance measure, introduce the PFLPWA operator, and investigate its fundamental properties and special cases.

Definition 11. Let $\alpha_i = \langle s_{\theta(\alpha_i)}, \mu_{\alpha_i}, v_{\alpha_i} \rangle$ $(i = 1, 2, \dots, n)$ be a collection of Pythagorean fuzzy linguistic numbers, and let $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of the array with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. The Pythagorean fuzzy linguistic power weighted average operator is defined as:

$$\text{PFLPWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{\sum_{i=1}^n w_i(1 + T(\alpha_i))\alpha_i}{\sum_{i=1}^n w_i(1 + T(\alpha_i))}$$

where $T(\alpha_i) = \sum_{j=1, j\neq i}^n \text{Supp}(\alpha_i, \alpha_j)$, and $\text{Supp}(\alpha_i, \alpha_j)$ is the support degree for α_i from α_j, calculated using the distance measure from Definition 10. The support degree satisfies: 1. $\text{Supp}(\alpha_i, \alpha_j) \in [0, 1]$; 2. $\text{Supp}(\alpha_i, \alpha_j) = \text{Supp}(\alpha_j, \alpha_i)$; 3. If $d(\alpha_i, \alpha_j) < d(\alpha_s, \alpha_t)$, then $\text{Supp}(\alpha_i, \alpha_j) \geq \text{Supp}(\alpha_s, \alpha_t)$.

Theorem 1. Let $\alpha_i = \langle s_{\theta(\alpha_i)}, \mu_{\alpha_i}, v_{\alpha_i} \rangle$ $(i = 1, 2, \dots, n)$ be Pythagorean fuzzy linguistic numbers with weight vector $w = (w_1, w_2, \dots, w_n)^T$ where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. The aggregated value using the PFLPWA operator is also a Pythagorean fuzzy linguistic number, given by:

$$\text{PFLPWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle s_{\sum_{i=1}^{n} u_i \theta(\alpha_i)}, \sqrt{1 - \prod_{i=1}^{n} (1 - \mu_{\alpha_i}^2)^{u_i}}, \prod_{i=1}^{n} v_{\alpha_i}^{u_i} \right\rangle$$

where $u_i = \frac{w_i(1+T(\alpha_i))}{\sum_{j=1}^{n} w_j(1+T(\alpha_j))}$.

Proof. The proof follows from Definition 11 and the operational rules in Definition 7 through mathematical induction.

Theorem 2. If the weight vector $w = (1/n, 1/n, \dots, 1/n)^T$, then the PFLPWA operator reduces to the Pythagorean fuzzy linguistic power average (PFLPA) operator:

$$\text{PFLPA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{\sum_{i=1}^{n} (1 + T(\alpha_i))\alpha_i}{\sum_{i=1}^{n} (1 + T(\alpha_i))}$$

Theorem 3. If $\text{Supp}(\alpha_i, \alpha_j) = k$ for all $i \neq j$, then the PFLPWA operator reduces to the Pythagorean fuzzy linguistic weighted average (PFLWA) operator:

$$\text{PFLWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{i=1}^{n} w_i \alpha_i$$

Properties of PFLPWA Operator:

Let $\alpha_i = \langle s_{\theta(\alpha_i)}, \mu_{\alpha_i}, v_{\alpha_i} \rangle$ and $\beta_i = \langle s_{\theta(\beta_i)}, \mu_{\beta_i}, v_{\beta_i} \rangle$ $(i = 1, 2, \dots, n)$ be two collections of Pythagorean fuzzy linguistic numbers.

a) **Commutativity.** If $(\alpha_1', \alpha_2', \dots, \alpha_n')$ is any permutation of $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then:

$$\text{PFLPWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \text{PFLPWA}_w(\alpha_1', \alpha_2', \dots, \alpha_n')$$

b) **Idempotency.** If $\alpha_i = \alpha$ for all i, then:

$$\text{PFLPWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha$$

c) **Boundedness.** Let $\alpha_{\min} = \langle s_{\min_i \theta(\alpha_i)}, \min_i \mu_{\alpha_i}, \max_i v_{\alpha_i} \rangle$ and $\alpha_{\max} = \langle s_{\max_i \theta(\alpha_i)}, \max_i \mu_{\alpha_i}, \min_i v_{\alpha_i} \rangle$. Then:

$$\alpha_{\min} \leq \text{PFLPWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha_{\max}$$

d) **Monotonicity.** If $\alpha_i \leq \beta_i$ for all i, then:

$$\text{PFLPWA}_w(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \text{PFLPWA}_w(\beta_1, \beta_2, \dots, \beta_n)$$

Proof of Monotonicity: Since $\alpha_i \leq \beta_i$, we have $\theta(\alpha_i) \leq \theta(\beta_i)$, $\mu_{\alpha_i} \leq \mu_{\beta_i}$, and $v_{\alpha_i} \geq v_{\beta_i}$. The weight u_i is the same for both aggregations. Therefore, the aggregated linguistic term, membership degree, and non-membership degree preserve the order, establishing the monotonicity property.

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3 Decision-Making Method Based on PFLPWA Operator

For multi-attribute decision-making problems in the Pythagorean fuzzy linguistic environment, let the expert group be $D = \{d_1, d_2, \dots, d_t\}$, the alternative set be $X = \{x_1, x_2, \dots, x_m\}$, and the attribute set be $C = \{c_1, c_2, \dots, c_n\}$. Assume the attribute weight vector is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. The expert weight vector is $e = (e_1, e_2, \dots, e_t)^T$ with $e_k \in [0, 1]$ and $\sum_{k=1}^{t} e_k = 1$. The decision matrix $U^{(k)} = (\alpha_{ij}^{(k)})_{m\times n}$ gives the Pythagorean fuzzy linguistic evaluation value of alternative x_i on attribute c_j by expert d_k, where $\alpha_{ij}^{(k)} = \langle s_{\theta(\alpha_{ij}^{(k)})}, \mu_{\alpha_{ij}^{(k)}}, v_{\alpha_{ij}^{(k)}} \rangle$, $s_{\theta(\alpha_{ij}^{(k)})} \in S$ is a linguistic term indicating the degree to which alternative x_i satisfies attribute c_j, with $\mu_{\alpha_{ij}^{(k)}}, v_{\alpha_{ij}^{(k)}} \in [0, 1]$ and $(\mu_{\alpha_{ij}^{(k)}})^2 + (v_{\alpha_{ij}^{(k)}})^2 \leq 1$.

The specific steps for solving Pythagorean fuzzy linguistic multi-attribute decision-making problems using the PFLPWA operator are as follows:

Step 1: Calculate support degrees.
For each expert d_k, compute the support degree between Pythagorean fuzzy linguistic numbers $\alpha_{ij}^{(k)}$ and $\alpha_{ij}^{(l)}$ $(k, l = 1, 2, \dots, t)$ using Definition 10:

$$\mathrm{Supp}(\alpha_{ij}^{(k)}, \alpha_{ij}^{(l)}) = 1 - d(\alpha_{ij}^{(k)}, \alpha_{ij}^{(l)})$$

Step 2: Compute comprehensive weights.
Based on the expert group weight vector e, calculate the support degree $T(\alpha_{ij}^{(k)})$ for each Pythagorean fuzzy linguistic number $\alpha_{ij}^{(k)}$:

$$T(\alpha_{ij}^{(k)}) = \sum_{l=1, l\neq k}^{t} e_l \cdot \mathrm{Supp}(\alpha_{ij}^{(k)}, \alpha_{ij}^{(l)})$$

Then compute the comprehensive weight $\xi_{ij}^{(k)}$ corresponding to $\alpha_{ij}^{(k)}$:

$$\xi_{ij}^{(k)} = \frac{e_k(1 + T(\alpha_{ij}^{(k)}))}{\sum_{l=1}^{t} e_l(1 + T(\alpha_{ij}^{(l)}))}$$

where $\xi_{ij}^{(k)} \geq 0$ and $\sum_{k=1}^{t} \xi_{ij}^{(k)} = 1$.

Step 3: Aggregate individual decision matrices.
Use the PFLPWA operator to aggregate individual decision matrices $U^{(k)}$ $(k = 1, 2, \ldots, t)$ and obtain the comprehensive decision matrix $U = (\alpha_{ij})_{m\times n}$:

$$\alpha_{ij} = \mathrm{PFLPWA}_e(\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \ldots, \alpha_{ij}^{(t)}) = \left\langle s_{\sum_{k=1}^{t} \xi_{ij}^{(k)} \theta(\alpha_{ij}^{(k)})}, \sqrt{1 - \prod_{k=1}^{t} (1 - (\mu_{\alpha_{ij}^{(k)}})^2)^{\xi_{ij}^{(k)}}}, \prod_{k=1}^{t} (v_{\alpha_{ij}^{(k)}})^{\xi_{ij}^{(k)}} \right\rangle$$

Step 4: Compute comprehensive attribute values.
Use the Pythagorean fuzzy linguistic weighted average (PFLWA) operator to aggregate the attribute values α_{ij} $(j = 1, 2, \ldots, n)$ for each alternative x_i and obtain the comprehensive attribute value α_i:

$$\alpha_i = \mathrm{PFLWA}_\omega(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) = \left\langle s_{\sum_{j=1}^{n} \omega_j \theta(\alpha_{ij})}, \sqrt{1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_{ij}}^2)^{\omega_j}}, \prod_{j=1}^{n} v_{\alpha_{ij}}^{\omega_j} \right\rangle$$

Step 5: Rank alternatives.
Calculate the score function values $s(\alpha_i)$ and accuracy function values $h(\alpha_i)$ for each comprehensive attribute value α_i using Definition 6, and rank the alternatives x_i $(i = 1, 2, \ldots, m)$ accordingly.

Step 6: Select the optimal alternative.
Based on the ranking results, select the best alternative.

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4 Example Analysis

Consider an investment company' s decision-making problem for selecting software projects [8]. The company evaluates alternatives from software industry feasibility perspectives, considering three attributes: technical feasibility (c_1), economic feasibility (c_2), and operational feasibility (c_3). The attribute weight vector is $\omega = (0.2, 0.5, 0.3)^T$. The expert group consists of three experts $D = \{d_1, d_2, d_3\}$ with weight vector $e = (1/3, 1/3, 1/3)^T$. The experts provide Pythagorean fuzzy linguistic evaluation matrices $U^{(k)} = (\alpha_{ij}^{(k)})_{4\times 3}$ $(k = 1, 2, 3)$ for four alternatives $X = \{x_1, x_2, x_3, x_4\}$, using the linguistic term set $S = \{s_0, s_1, \ldots, s_6\} = \{$极差, 很差, 差, 一般, 好, 很好, 极好$\}$.

The individual evaluation matrices are:

$$U^{(1)} = \begin{pmatrix} \langle s_2, 0.8, 0.6\rangle & \langle s_3, 0.7, 0.3\rangle & \langle s_4, 0.6, 0.2\rangle \\ \langle s_3, 0.7, 0.6\rangle & \langle s_4, 0.8, 0.3\rangle & \langle s_2, 0.6, 0.4\rangle \\ \langle s_2, 0.6, 0.2\rangle & \langle s_3, 0.8, 0.2\rangle & \langle s_4, 0.7, 0.4\rangle \\ \langle s_1, 0.7, 0.5\rangle & \langle s_3, 0.9, 0.2\rangle & \langle s_2, 0.7, 0.4\rangle \end{pmatrix}$$

$$U^{(2)} = \begin{pmatrix} \langle s_4, 0.7, 0.6\rangle & \langle s_2, 0.7, 0.6\rangle & \langle s_3, 0.6, 0.4\rangle \\ \langle s_2, 0.8, 0.3\rangle & \langle s_3, 0.7, 0.3\rangle & \langle s_4, 0.6, 0.3\rangle \\ \langle s_3, 0.6, 0.2\rangle & \langle s_2, 0.6, 0.4\rangle & \langle s_3, 0.7, 0.3\rangle \\ \langle s_4, 0.7, 0.3\rangle & \langle s_3, 0.8, 0.5\rangle & \langle s_2, 0.8, 0.2\rangle \end{pmatrix}$$

$$U^{(3)} = \begin{pmatrix} \langle s_3, 0.5, 0.6\rangle & \langle s_2, 0.6, 0.3\rangle & \langle s_6, 0.6, 0.4\rangle \\ \langle s_4, 0.5, 0.6\rangle & \langle s_3, 0.6, 0.3\rangle & \langle s_2, 0.8, 0.2\rangle \\ \langle s_3, 0.5, 0.2\rangle & \langle s_4, 0.8, 0.4\rangle & \langle s_3, 0.5, 0.2\rangle \\ \langle s_2, 0.7, 0.5\rangle & \langle s_3, 0.7, 0.6\rangle & \langle s_4, 0.7, 0.2\rangle \end{pmatrix}$$

Step 1: Calculate support degrees $\mathrm{Supp}(\alpha_{ij}^{(k)}, \alpha_{ij}^{(l)})$ using Definition 10.

Step 2: Compute comprehensive weights $\xi_{ij}^{(k)}$:

$$\xi^{(1)} = \begin{pmatrix} 0.3283 & 0.3347 & 0.3389 \\ 0.3514 & 0.3230 & 0.3412 \\ 0.3219 & 0.3358 & 0.3377 \\ 0.3345 & 0.3274 & 0.3364 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 0.3383 & 0.3282 & 0.3359 \\ 0.3174 & 0.3425 & 0.3317 \\ 0.3424 & 0.3358 & 0.3441 \\ 0.3368 & 0.3351 & 0.3367 \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 0.3334 & 0.3371 & 0.3252 \\ 0.3312 & 0.3345 & 0.3271 \\ 0.3357 & 0.3285 & 0.3182 \\ 0.3288 & 0.3375 & 0.3269 \end{pmatrix}$$

Step 3: Aggregate individual matrices using PFLPWA to obtain the comprehensive decision matrix U:

$$U = \begin{pmatrix} \langle s_{3.01}, 0.69, 0.60\rangle & \langle s_{2.33}, 0.67, 0.38\rangle & \langle s_{4.31}, 0.60, 0.32\rangle \\ \langle s_{2.70}, 0.69, 0.48\rangle & \langle s_{3.32}, 0.71, 0.30\rangle & \langle s_{2.33}, 0.65, 0.23\rangle \\ \langle s_{2.64}, 0.57, 0.20\rangle & \langle s_{3.00}, 0.69, 0.40\rangle & \langle s_{3.32}, 0.70, 0.23\rangle \\ \langle s_{3.00}, 0.67, 0.43\rangle & \langle s_{1.67}, 0.74, 0.53\rangle & \langle s_{4.01}, 0.82, 0.20\rangle \end{pmatrix}$$

Step 4: Compute comprehensive attribute values α_i for each alternative using PFLWA:

$$\alpha_1 = \langle s_{3.06}, 0.66, 0.39\rangle, \quad \alpha_2 = \langle s_{2.90}, 0.69, 0.30\rangle$$

$$\alpha_3 = \langle s_{3.02}, 0.67, 0.30\rangle, \quad \alpha_4 = \langle s_{2.64}, 0.76, 0.38\rangle$$

Step 5: Calculate score function values $s(\alpha_i)$:

$$s(\alpha_1) = 0.1416, \quad s(\alpha_2) = 0.1816, \quad s(\alpha_3) = 0.1827, \quad s(\alpha_4) = 0.1885$$

Step 6: Rank alternatives based on score values:

$$\alpha_4 > \alpha_3 > \alpha_2 > \alpha_1$$

Therefore, the optimal alternative is x_4.

This example demonstrates that during the expert evaluation phase, the sum of membership and non-membership degrees often exceeds 1, which is not permitted for intuitionistic fuzzy numbers. However, Pythagorean fuzzy numbers accommodate such cases reasonably, reflecting a more relaxed evaluation range. Simultaneously, experts can reference linguistic term sets during assessment, providing both referential value and standardized scaling. Furthermore, during the fusion of expert evaluation information, the PFLPWA operator assigns each Pythagorean fuzzy linguistic number $\alpha_{ij}^{(k)}$ a weight $\xi_{ij}^{(k)}$ that combines subjective weights e_k with support degrees. The support degree mechanism reduces the weight of evaluations that deviate significantly from group opinions, preventing excessive dependence on decision-makers' subjective consciousness and enabling flexible handling of group decision-making problems. When expert group weights are unknown, equal weights can be assigned, causing the PFLPWA operator to degenerate into the PFLPA operator.

5 Conclusion

Pythagorean fuzzy linguistic sets extend intuitionistic fuzzy linguistic sets by combining linguistic sets with Pythagorean fuzzy sets, enabling evaluators to conveniently reference linguistic information while providing more flexible value ranges. To handle multi-attribute decision-making problems using Pythagorean fuzzy linguistic sets, this paper constructs a distance measure for Pythagorean fuzzy linguistic numbers, defines support degrees between them, and proposes the Pythagorean fuzzy linguistic power weighted average (PFLPWA) operator along with a corresponding group decision-making algorithm. The operator's weights incorporate both subjectively provided expert weights and objective mutual support degrees among data, addressing evaluation differences among experts. The proposed distance measure and PFLPWA operator enrich theoretical research in this domain, and the resulting group decision-making algorithm offers referential evaluation values, flexible value assignment, and consideration of evaluation differences compared to traditional methods.

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Note: Figure translations are in progress. See original paper for figures.