Lyapunov-type inequalities for —Laplacian equations

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Abstract

In this work, we present several Lyapunov-type inequalities for a class of $\psi-$Laplacian equations of the form

$$(\psi(u'(x)))' + r(x)f(u(x)) = 0,$$

with Dirichlet boundary conditions, where ψ and f satisfy certain structural conditions with general nonlinearities. We do not require any sub-multiplicative property of ψ, and any convexity of $\frac{1}{\psi(t)}$ or $\psi(t)t$ in the establishment of Lyapunov-type inequalities. The obtained inequalities can be seen as extensions and complements of the existing results in the literature.

Full Text

Preamble

Lyapunov-type inequalities for -Laplacian equations

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Abstract

In this work, we present several Lyapunov-type inequalities for a class of -Laplacian equations of the form $(\psi(u'(x)))' + r(x)f(u(x)) = 0$, with Dirichlet boundary conditions, where ψ and f satisfy certain structural conditions with general nonlinearities. We do not require any sub-multiplicative property of ψ,

nor any convexity of $\psi(t)$ or $\psi(t)t$ in establishing these Lyapunov-type inequalities. The obtained results can be seen as extensions and complements of existing results in the literature.

Key words: Lyapunov inequality, -Laplacian, nonlinear equation.

Introduction

Consider the Hill' s equation $u''(x) + r(x)u(x) = 0$, $u(a) = u(b) = 0$, $x \in (a, b)$, where r is a continuous and nonnegative function defined on $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. If there exists a nontrivial solution u of this equation, then the inequality $\int_a^b r(x)dx \geq \frac{4}{b-a}$ holds. This result is due originally to Lyapunov [**?**] and is known as the "Lyapunov inequality." The Lyapunov inequality and its many generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theories of differential and difference equations, as well as in time scales.

In recent years, independent works have appeared generalizing Lyapunov' s inequality for the p-Laplacian, using Hölder, Jensen, or Cauchy-Schwarz inequalities. A thorough literature review of Lyapunov-type inequalities and their applications can be found in the survey articles by Brown and Hinton [**?**], Cheng [**?**], and Tiryaki [**?**]. Other related topics can be found in recent articles [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] and the references therein.

We now present some results directly related to our problem. In 2005, De Nápoli and Pinasco considered Lyapunov-type inequalities for certain nonlinear differential equations (-Laplacian equations) generalizing the p-Laplacian. The main result in [**?**] is:

Theorem A Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing function such that $\psi(t) = t\psi(t)$ is a convex function. Moreover, suppose that there exists a constant $k > 0$ such that $\psi(2t) \leq k\psi(t)$ for any $t \geq 0$. If $r(x)$ is a positive integrable function, and the following problem

$$(\psi(u'(x)))' + r(x)\psi(u(x)) = 0 \text{ in } (a, b), \quad u(a) = u(b) = 0$$

admits a nontrivial solution, then

$$\int_a^b r(x)dx \geq \frac{2}{k} \cdot \frac{1}{[1 - \log_2(b-a)]}$$

where $[v]$ is the largest integer less than or equal to v.

In 2011, Sánchez and Vergara extended this result to equations with a general nonlinear form, considering (see [**?**])

$$(\psi(u'(x)))' + \lambda r(x)f(u(x)) = 0 \text{ in } (a, b), \quad u(a) = u(b) = 0,$$

where $\lambda > 0$ is a constant. Under the following assumptions: - (B1) $f \in C(\mathbb{R})$ is odd and satisfies $tf(t) > 0$ for $t \neq 0$. - (B2) $r : [a, b] \to (0, +\infty)$ is a continuous function. - (B3) ψ is odd, increasing, and sub-multiplicative on $[0,+\infty)$, and $\psi(t)$ is convex in $t>0$.

The authors established a Lyapunov-type inequality for this problem, with the following result:

Theorem B Suppose that conditions (B1)-(B3) are satisfied. If u is a nontrivial solution of problem (4), satisfying $u(x)\neq 0$ for $x\in(a,b)$, then the following inequality holds:

$$\psi\left(\frac{2}{b-a}\right)\leq\lambda\int_a^b\frac{f(u(x))}{\psi(u(x))}dx$$

when the integral exists.

The convexity of $t\psi(t)$ (or $\psi(t)$) and the sub-multiplicative property of ψ play essential roles in establishing Lyapunov-type inequalities in [**?**] (or [**?**]). Our motivation for this paper comes from the works [**?**] and [**?**]. The main novelty of this paper is to establish Lyapunov-type inequalities for a large class of nonlinear equations governed by (5) (or (4)) without requiring any convexity assumption on $t\psi(t)$ or $\psi(t)$, and without any sub-multiplicative assumption on ψ. The function ψ in this paper permits much more general nonlinearities than those in [**?**, **?**] (see, e.g., Remark 2 in Section 1 and examples in Section 4). Moreover, under assumption (H4), we do not require any odd-even properties of f, and require less restrictive sign conditions on f than those in [**?**].

The rest of this paper is organized as follows. Section 2 presents the problem under consideration and the main results on Lyapunov-type inequalities, along with some remarks on the structural conditions. Detailed proofs of the Lyapunov-type inequalities are given in Section 3. Additional examples satisfying our structural conditions are provided in Section 4.

2 Problem Setting and Main Results

In this paper, we establish Lyapunov-type inequalities for the following equation:

$$(\psi(u'(x)))'+r(x)f(u(x))=0\text{ in }(a,b),\quad u(a)=u(b)=0,$$

where ψ and f satisfy the following structural conditions with general nonlinearities:

- (H1) $\psi,f\in C((-\infty,\infty))\cap C^1((0,\infty))$ with $f\not\equiv 0$ on $(-\infty,\infty)$.
- (H2) ψ is odd on $(-\infty,\infty)$.
- (H3) $f(t)\geq 0$ for all $t\in[0,\infty)$.
- (H4) There exists $k_0>0$ such that $|f(t)|\leq k_0\psi(|t|)$ for all $t\in(-\infty,\infty)$.

We make further assumptions on ψ or f:

- (H) There exist constants $\delta_0,\delta_1\geq 0$ such that $\delta_0\psi(t)\leq t\psi'(t)\leq\delta_1\psi(t)$, $\forall t>0$.

- (Hf) There exist constants $\theta_0, \theta_1 \geq 0$ such that $\theta_{0f}(t) \leq tf'(t) \leq \theta_{1f}(t)$, $\forall t > 0$.

Throughout this paper, we always assume that $r \in L^1(a,b)$ with $r \not\equiv 0$ on (a,b), and conditions (H1)-(H4) are satisfied. Moreover, we always assume that (5) has a non-trivial solution u in the sense that $u \in C^1(a,b) \cap C([a,b])$, $\psi(u'(x))$ is absolutely continuous in x, and u satisfies the equation in (5) almost everywhere in (a,b).

The first main result is as follows, which can be seen as a complement to the works of [**?**] and [**?**] for functions satisfying (H) or (Hf).

Theorem 1 - (ii) If f satisfies (Hf), then $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\theta_0}{1+\theta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$. - (i) If ψ satisfies (H), then $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \frac{1+\delta_0}{1+\delta_1} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$.

We present some corollaries of Theorem 1.

Corollary 2 - (i) If $\psi(t) = f(t) = |t|^{p-2}t$ $(p > 1)$, then $\int_a^b |r(x)|dx \geq \frac{2^p}{(b-a)^{p-1}}$, which is one of the results obtained independently in [**?**, **?**]. - (ii) If $\psi(t) = f(t) = |t|^{a-1}t \log^c(b|t| + d)$, $a, b > 0$, $c, d > 1$, then $\int_a^b |r(x)|dx \geq \frac{2(1+a\ln d)}{1+(1+a)\ln d} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a}, \left(\frac{2}{b-a}\right)^{a+\frac{1}{\ln d}}\right\}$. - (iii) If $\psi(t) = f(t) = \frac{|t|^{a-1}t}{\log^c(b|t|+d)}$, $b > 0$, $c, d > 1$, $a > \frac{1}{\ln d}$, then $\int_a^b |r(x)|dx \geq \frac{2(1+a\ln d)}{1+(1+a)\ln d} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{a-\frac{1}{\ln d}}, \left(\frac{2}{b-a}\right)^{a}\right\}$.

If we make the further assumption that $\psi(t)t$ (or $f(t)t$) is convex on $[0, +\infty)$, we obtain stronger results than Theorem 1, which can be seen as extensions of [**?**].

Theorem 3 Assume further that $\psi(t)t$ is convex in $t \in [0, +\infty)$. - (i) If ψ satisfies (H), then $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\delta_0}, \left(\frac{2}{b-a}\right)^{\delta_1}\right\}$. - (ii) If f satisfies (Hf), then $\int_a^b |r(x)|dx \geq \frac{2}{k_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\theta_1}\right\}$.

Before proving the main results, we give some remarks on the structural conditions.

Remark 1 - (i) We point out that (H) (or (Hf)) is a slight variation of Lieberman' s condition in [**?**], where regularity theory was considered for a class of elliptic partial differential equations with a structural condition described by $0 < \delta_0 \leq \frac{t\psi'(t)}{\psi(t)} \leq \delta_1$. It should be noted that $t\psi'(t)$ (or $tf'(t)$) in (H) (or (Hf)) can be zero at some point $t_0 > 0$, i.e., $\delta_0 = 0$ (or $\theta_0 = 0$). Indeed, considering $\psi(t) = (t-1)^3 + 1$, we have $\psi \in C^1((0, +\infty))$ and $0 \leq \frac{t\psi'(t)}{\psi(t)} \leq 3$ for all $t > 0$. The lower boundedness can be achieved when $\psi'(1) = 3(t-1)^2|_{t=1} = 0$. - (ii) By (H1)-(H4), $\psi(0) = f(0) = 0$ and $\psi(t) \geq 0$ for any $t \geq 0$. Furthermore, if ψ (or f) satisfies (H) (or (Hf)), then $\psi'(t) \geq 0$ (or $f'(t) \geq 0$), which guarantees the increasing monotonicity of $\psi(t)$ (or $f(t)$) for $t \geq 0$.

Remark 2 In this remark, we provide two examples of the function ψ (or f) showing that our assumptions on ψ (or f) are much weaker than those in [**?**, **?**] (see Theorems A and B) in a certain sense. More examples of functions satisfying (H) or (Hf) are provided in Section 4. - (i) Let $\psi_0(t) = t + \frac{1}{(t+1)^3}$ with $t \in [\frac{1}{6}, 3]$. Due to the continuity of ψ_0 and ψ_0', there exist $\delta_0', \delta_1' > 0$ such that $\delta_0' \leq \frac{t\psi_0'(t)}{\psi_0(t)} \leq \delta_1'$ for all $t \in [\frac{1}{6}, 3]$. Let $p = \frac{11}{6}$, $a = (\frac{6}{5} + \frac{5}{6})^{-p}$, and $q = \frac{1}{2}$, $b = (\sqrt{3} + \frac{1}{\sqrt{3}})^{-q}$. Define

$$\psi(t) = \begin{cases} at^p, & 0 < t < \frac{1}{6} \\ \psi_0(t), & \frac{1}{6} \leq t \leq 3 \\ bt^q, & t > 3 \end{cases}$$

Thus $\psi \in C^1((0, +\infty))$ and $\delta_0' \leq \frac{t\psi'(t)}{\psi(t)} \leq \delta_1'$ for all $t \in [\frac{1}{6}, 3]$. By direct computation, one may verify that $(\psi(t)t)'' < 0$ for $t \in (\frac{1}{6}, 3)$, i.e., $\psi(t)t$ is not convex on $[0, +\infty)$, which means ψ does not satisfy condition (H3) in [**?**]. However, ψ defined as above satisfies (H) with $\delta_0 = \min\{\delta_0', p, q\}$ and $\delta_1 = \max\{\delta_1', p, q\}$ in this paper. - (ii) Let $\psi_0(t) = \sin t$ with $t \in (0, \frac{\pi}{2})$. Then $(\psi_0(t)t)'' = -t\sin t + 2\cos t \to -\frac{\pi}{2} < 0$ as $t \to (\frac{\pi}{2})^-$. Thus there exists $[t_0, t_1] \subset (0, \frac{\pi}{2})$ such that $(\psi_0(t)t)'' < 0$ for all $t \in [t_0, t_1]$. Let

$$\psi(t) = \begin{cases} at^p, & 0 < t < t_0 \\ \psi_0(t), & t_0 \leq t \leq t_1 \\ bt^q, & t > t_1 \end{cases}$$

where $p = t_0 \cot t_0 > 0$, $a = t_0^{-p} \sin t_0 > 0$ and $q = t_1 \cot t_1 > 0$, $b = t_1^{-q} \sin t_1 > 0$. Note that there exists $\delta_0', \delta_1' > 0$ such that $\delta_0' \leq \frac{t\psi'(t)}{\psi(t)} \leq \delta_1'$ for all $t \in [t_0, t_1]$. One may verify that ψ satisfies (H) with $\delta_0 = \min\{\delta_0', p, q\}$ and $\delta_1 = \max\{\delta_1', p, q\}$. However, $(\psi(t)t)'' < 0$ for $t \in [t_0, t_1]$, which means such a ψ does not satisfy the convexity condition in [**?**] while it still satisfies (H) in this paper.

3 Proof of Main Results

We first present some auxiliary results needed in the main proof. Let $\Psi(t) = \int_0^t \psi(s)ds$ for $t \geq 0$.

Lemma 4 Assume that ψ satisfies (H1)-(H4) and (H). The following results hold true: - (i) $\psi(st) \leq \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t)$, $\forall s, t \geq 0$. - (ii) Ψ is C^2-continuous on $(0, +\infty)$, and convex on $[0, +\infty)$. - (iii) $\frac{t\psi(t)}{1+\delta_1} \leq \Psi(t) \leq \frac{t\psi(t)}{1+\delta_0}$, $\forall t \geq 0$.

Proof. Let $h_0(t) = \frac{\psi(t)}{t^{\delta_0}}$ and $h_1(t) = \frac{\psi(t)}{t^{\delta_1}}$ for $t > 0$. By (H), it follows that

$$h_0'(t) = \frac{\psi'(t)t^{\delta_0} - \psi(t)\delta_{0t}^{\delta_0 - 1}}{t^{2\delta_0}} = \frac{t\psi'(t) - \psi(t)\delta_0}{t^{\delta_0+1}} \geq 0$$

which implies that $h_0(t)$ is increasing for $t > 0$. Therefore $h_0(st) \le h_0(t)$ for $0 \le s \le 1$. It follows that

$$\psi(st) \le s^{\delta_0}\psi(t), \quad \forall t > 0, 0 \le s \le 1.$$

Similarly, one may prove that $h_1(t)$ is decreasing for $t > 0$. Then $h_1(st) \le h_1(t)$ for $s \ge 1$. It follows that

$$\psi(st) \le s^{\delta_1}\psi(t), \quad \forall t > 0, s \ge 1.$$

By these two inequalities, we have

$$\psi(st) \le \max\{s^{\delta_0}, s^{\delta_1}\}\psi(t), \quad \forall t > 0, s \ge 0,$$

which together with the continuity of ψ at $t = 0$ yields (i). (ii) is obvious since $\Psi''(t) = \psi'(t) \ge 0$ for $t > 0$ (see Remark 1) and $\Psi(t)$ is continuous at $t = 0$.

To prove (iii), let $\Psi_0(t) = (1+\delta_0)\Psi(t) - t\psi(t)$ and $\Psi_1(t) = (1+\delta_1)\Psi(t) - t\psi(t)$ for $t \ge 0$. It is easy to see that $\Psi_0'(t) \le 0$ and $\Psi_1'(t) \ge 0$ for $t > 0$. Then $\Psi_0(t) \le \Psi_0(0) = 0$ and $\Psi_1(t) \ge \Psi_1(0) = 0$, which together with the continuity of Ψ_0, Ψ_1 yields (iii). □

Remark 3 Let $F(t) = \int_0^t f(s)ds$ for $t \ge 0$. Then the function f satisfying (H1)-(H4) and (Hf) and the function F have similar properties as above.

Proof of Theorem 1. Without loss of generality, assume that $|u(c)| = \max_{x\in[a,b]} |u(x)| > 0$ with $c \in (a, b)$. Note that

$$|u(c)| = \left|\int_a^c u'(x)dx\right| = \left|\int_a^c u'(x)dx - \int_c^b u'(x)dx\right| \le \int_a^c |u'(x)|dx + \int_c^b |u'(x)|dx = \int_a^b |u'(x)|dx.$$

Firstly, we prove Theorem 1(i) under the assumption that ψ satisfies the structural condition (H). Indeed, by the monotonicity of ψ, and Lemma 4(i) and (iii), we get

$$\psi(|u(c)|)|u(c)| \le \psi\left(\int_a^b |u'(x)|dx\right)\int_a^b |u'(x)|dx$$

$$\le \max\left\{\left(\frac{1}{b-a}\right)^{\delta_0}, \left(\frac{1}{b-a}\right)^{\delta_1}\right\}\psi\left(\int_a^b |u'(x)|dx\right)\int_a^b |u'(x)|dx$$

$$\le (1+\delta_1)\cdot\max\left\{\left(\frac{1}{b-a}\right)^{\delta_0}, \left(\frac{1}{b-a}\right)^{\delta_1}\right\}\Psi\left(\int_a^b |u'(x)|dx\right)$$

$$\le (1+\delta_1)\cdot\max\left\{\left(\frac{1}{b-a}\right)^{\delta_0}, \left(\frac{1}{b-a}\right)^{\delta_1}\right\}\cdot\frac{1}{b-a}\int_a^b \Psi(|u'(x)|)dx$$

$$= (1+\delta_1) \cdot \max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}}, \frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\} \int_a^b \Psi(|u'(x)|)dx,$$

where in the last inequality we used the convexity of Ψ (see Lemma 4(ii)).

Multiplying (5) by u, integrating over (a,b), and using Lemma 4(iii), (H2)-(H4), and the inequality above, we get

$$\int_a^b \Psi(|u'|)dx \le \frac{1}{1+\delta_0}\int_a^b \psi(|u'|)|u'|dx = \frac{1}{1+\delta_0}\int_a^b \psi(u')u'dx$$

$$= \frac{1}{1+\delta_0}\int_a^b r(x)f(u)udx \le \frac{1}{1+\delta_0}\int_a^b |r(x)||f(u)u|dx$$

$$\le \frac{k_0}{1+\delta_0}\max_{x\in[a,b]}(\psi(|u|)|u|)\int_a^b |r(x)|dx = \frac{k_0}{1+\delta_0}\psi(|u(c)|)|u(c)|\int_a^b |r(x)|dx$$

$$\le \frac{k_0(1+\delta_1)}{1+\delta_0}\cdot \max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}}, \frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\int_a^b \Psi(|u'|)dx \cdot \int_a^b |r(x)|dx.$$

Note that $\int_a^b \Psi(|u'|)dx > 0$; otherwise, if $\int_a^b \Psi(|u'|)dx = 0$, then by the inequality above and Lemma 4(i), we would have

$$0 \le \psi(t)|u(c)| = \psi\left(t\cdot\frac{|u(c)|}{|u(c)|}\right)|u(c)| \le \max\left\{\left(\frac{t}{|u(c)|}\right)^{\delta_0}, \left(\frac{t}{|u(c)|}\right)^{\delta_1}\right\}\psi(u(c))|u(c)| \le 0, \quad \forall t \ge 0,$$

which implies $\psi \equiv 0$ for all $t \in [0,+\infty)$. Then by the odd property of ψ, we have $\psi \equiv 0$ for all $t \in (-\infty,+\infty)$. Due to (H4), $f \equiv 0$ for all $t \in (-\infty,+\infty)$, which contradicts assumption (H1).

Therefore, we obtain

$$\int_a^b |r(x)|dx \ge \frac{1+\delta_0}{k_0(1+\delta_1)}\cdot \min\left\{\frac{2^{1+\delta_0}}{(b-a)^{\delta_0}}, \frac{2^{1+\delta_1}}{(b-a)^{\delta_1}}\right\}.$$

Thus Theorem 1(i) is proven.

Now for Theorem 1(ii), we proceed in a similar manner. Indeed, if f satisfies the structural condition (Hf), proceeding as above, we get

$$f(|u(c)|)|u(c)| \le (1+\theta_1)\cdot \max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}}, \frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\}\int_a^b F(|u'(x)|)dx.$$

Multiplying (5) by u, integrating over (a,b), and using Lemma 4(iii), (H2)-(H4), and the inequality above, we get

$$\int_a^b F(|u'|)dx \le \frac{1}{1+\theta_0}\int_a^b f(|u'|)|u'|dx \le \frac{k_0}{1+\theta_0}\int_a^b \psi(|u'|)|u'|dx$$

$$= \frac{k_0}{1+\theta_0}\int_a^b \psi(u')u'dx = \frac{k_0}{1+\theta_0}\int_a^b r(x)f(u)udx$$

$$\leq \frac{k_0}{1+\theta_0}\max_{x\in[a,b]}(|f(u)u|)\int_a^b |r(x)|dx = \frac{k_0}{1+\theta_0}f(|u(c)|)|u(c)|\int_a^b |r(x)|dx$$

$$\leq \frac{k_0(1+\theta_1)}{1+\theta_0}\cdot\max\left\{\frac{(b-a)^{\theta_0}}{2^{1+\theta_0}},\frac{(b-a)^{\theta_1}}{2^{1+\theta_1}}\right\}\int_a^b F(|u'|)dx\cdot\int_a^b |r(x)|dx.$$

Note that $\int_a^b F(|u'|)dx > 0$; otherwise, we can argue as in the proof of (i) to conclude $f(t) \equiv 0$ for any $t \in (-\infty, +\infty)$. Finally, the inequality above implies the desired result. □

Proof of Theorem 3. The proof of Theorem 3 is a slight modification of the proof of Theorem 1. Indeed, let $\Phi(t) = \psi(t)t$ for $t \geq 0$, and let $c, u(c)$ be defined as in the proof of Theorem 1. If ψ satisfies the structural condition (H), arguing as before, we get

$$\psi(|u(c)|)|u(c)| \leq \max\left\{\left(\frac{1}{b-a}\right)^{\delta_0},\left(\frac{1}{b-a}\right)^{\delta_1}\right\}\Phi\left(\int_a^b |u'|dx\right)$$

$$\leq \max\left\{\left(\frac{1}{b-a}\right)^{\delta_0},\left(\frac{1}{b-a}\right)^{\delta_1}\right\}\cdot\frac{1}{b-a}\int_a^b \Phi(|u'|)dx$$

$$= \max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}},\frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\int_a^b \Phi(|u'|)dx,$$

where in the last inequality we used the convexity of Φ.

Combining this with (5), we have

$$\int_a^b \Phi(|u'|)dx = \int_a^b \psi(|u'|)|u'|dx = \int_a^b \psi(u')u'dx = \int_a^b r(x)f(u)udx$$

$$\leq k_0\psi(|u(c)|)|u(c)|\int_a^b |r(x)|dx \leq k_0\cdot\max\left\{\frac{(b-a)^{\delta_0}}{2^{1+\delta_0}},\frac{(b-a)^{\delta_1}}{2^{1+\delta_1}}\right\}\int_a^b \Phi(|u'|)dx\cdot\int_a^b |r(x)|dx,$$

which yields the desired result in Theorem 3(i). The desired result in Theorem 3(ii) can be proven in a similar way. □

Proof of Corollary 2. For (i), it should be noted that $\delta_0 = \theta_0 = \delta_1 = \theta_1 = p-1$ in (H) and (Hf).

For (ii), it should be noted that for $t \geq 0$, $\psi(t) = f(t) = t^a \log^c(bt+d)$, $a, b > 0$, $c, d > 1$. Then we have

$$\frac{t\psi'(t)}{\psi(t)} = a + \frac{bt}{(bt+d)\ln c}\cdot\log^c(bt+d) \leq a + \frac{bt}{(bt+d)\ln c}\cdot\log^c d = a + \frac{1}{\ln d}, \quad \forall t > 0.$$

Note that $\log^c(bt+d) \le \log^c d$ for all $t>0$, it follows that

$$a \le \frac{t\psi'(t)}{\psi(t)} \le a + \frac{1}{\ln d}, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = a > 0$, $\delta_1 = \theta_1 = a + \frac{1}{\ln d} > 0$ in (H) and (Hf).

For (iii), it should be noted that for $t \ge 0$, $\psi(t) = f(t) = \frac{t^a}{\log^c(bt+d)}$, $b > 0$, $c, d > 1$, $a > \frac{1}{\ln d}$. Then we have

$$\frac{t\psi'(t)}{\psi(t)} = a - \frac{bt}{(bt+d)\ln(bt+d)} \le a, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = a - \frac{1}{\ln d} > 0$, $\delta_1 = \theta_1 = a > 0$ in (H) and (Hf). □

4 Examples

In this section, we provide additional examples of $\psi(t)$ (or $f(t)$) satisfying (H) (or (Hf)), and determine the corresponding δ_0, δ_1 (or θ_0, θ_1). For simplicity, we restrict $\psi(t)$ (or $f(t)$) to the case $t \in [0, +\infty)$, since one may construct functions by odd or even extensions to $t \in (-\infty, +\infty)$.

Example 1 $\psi(t) = f(t) = \ln(1+at) + bt$, $\forall a > 0, b > 0$.

For this example, we have

$$\frac{t\psi'(t)}{\psi(t)} = \frac{\frac{at}{1+at} + bt}{\ln(1+at) + bt} \le \frac{a+b}{\ln(1+at)+bt} + 1, \quad \forall t > 0.$$

Note that $\ln(1+at) \le at$ for all $t > 0$, it follows that

$$\frac{a}{a+b} \le \frac{t\psi'(t)}{\psi(t)} \le \frac{a+b}{b} + 1, \quad \forall t > 0.$$

Thus $\delta_0 = \theta_0 = \frac{a}{a+b} > 0$, $\delta_1 = \theta_1 = \frac{a+b}{b} + 1 > 0$ in (H) and (Hf).

Example 2 $\psi(t) = f(t) = (1+t)\ln(1+t) - t$.

For this example, firstly note that $\psi'(t) = \ln(1+t) \ge 0$ for any $t \ge 0$. Thus $\psi(t) \ge \psi(0) = 0$. By direct computation, we have

$$\frac{t\psi'(t)}{\psi(t)} = \frac{t\ln(1+t)}{(1+t)\ln(1+t) - t} = \frac{t}{(1+t) - \frac{t}{\ln(1+t)}}, \quad \forall t > 0.$$

Since $\ln(1+t) \le t$ for all $t > 0$, it follows that

$$\frac{t\psi'(t)}{\psi(t)} \le \frac{t}{(1+t) - t} = 1, \quad \forall t > 0.$$

In the following, we prove that for any $t > 0$, there holds

$$\frac{t\psi'(t)}{\psi(t)} \geq \frac{1}{2}.$$

Indeed, let $h_1(t) = t\ln(1+t) - 2((1+t)\ln(1+t) - t) = 2t - t\ln(1+t) - 2\ln(1+t)$. Then $h_1'(t) = 2 - \ln(1+t) - \frac{t}{1+t}$. Let $h_2(t) = (1+t) - (1+t)\ln(1+t) - 1 = t - (1+t)\ln(1+t)$. It is easy to check that $h_2'(t) = -\ln(1+t) < 0$ for any $t > 0$. Thus $h_2(t) \leq h_2(0) = 0$, which leads to $h_1'(t) \leq 0$ for any $t > 0$. Therefore $h_1(t) \leq h_1(0) = 0$. As a consequence, the inequality above holds true for any $t > 0$. Finally, $\delta_0 = \theta_0 = \frac{1}{2}$, $\delta_1 = \theta_1 = 1$ in (H) and (Hf). □

Example 3

$$\psi(t) = f(t) = \begin{cases} at^p, & 0 \leq t < t_0, \\ bt^q + c, & t \geq t_0, \end{cases}$$

where $a, b, c, p, q, t_0 > 0$ such that $at_0^p = bt_0^q + c$, and $apt_0^{p-1} = bqt_0^{q-1}$.

For this example, we have $\psi = f \in C^1((0, +\infty))$ and $\min\{p, q\} \leq \frac{t\psi'(t)}{\psi(t)} \leq \max\{p, q\}$. Thus $\delta_0 = \theta_0 = \min\{p, q\} > 0$, $\delta_1 = \theta_1 = \max\{p, q\} > 0$ in (H) and (Hf).

Example 4 The following example is interesting since ψ or f has a variable exponent:

$$\psi(t) = f(t) = \begin{cases} at^p, & 0 \leq t < t_0, \\ bt^{g(t)-1}, & t \geq t_0, \end{cases}$$

where $t_0 > 1$, $a, b, p > 0$, and the function $g \in C^1([t_0, +\infty))$ satisfies

$$c \leq g'(t)t\ln t + g(t) - 1 \leq d, \quad \forall t \geq t_0,$$

with some constants $d \geq c > 0$. Note that

$$\frac{t(bt^{g(t)-1})'}{bt^{g(t)-1}} = tg'(t)\ln t + g(t) - 1.$$

By direct computation, one may verify that $\psi = f \in C^1((0, +\infty))$ and satisfies (H) and (Hf) with $\delta_0 = \theta_0 = \min\{p, c\} = c$, $\delta_1 = \theta_1 = \max\{p, d\} = d$. □

Example 5 In [**?**], the authors provided two examples of $\psi(t)$, i.e., (i) $\psi(t) = |t|^a\phi_p(t)$ with $a > 1 - p$, and (ii) $\psi(t) = (\ln(|t| + b))^b\phi_p(t)$ with $a \geq e$, $b > 0$, showing that these satisfy the structural condition (H). Indeed, for (i), it is easy to see that $\frac{t\psi'(t)}{\psi(t)} = a + p - 1 > 0$ for $t > 0$. Thus $\delta_0 = \delta_1 = a + p - 1$ in (H). For (ii), by direct computation, we have

$$\frac{t\psi'(t)}{\psi(t)} = p + \frac{bt}{(t+a)\ln(t+a)}, \quad t > 0.$$

Then $\delta_0 = p$, $\delta_1 = p + \frac{b}{\ln a}$ in (H). Note that $0 \leq \frac{bt}{(t+a)\ln(t+a)} \leq \frac{b}{\ln a}$ for $t > 0$. □

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Note: Figure translations are in progress. See original paper for figures.