Energy Optimization of Shape-Parameterized Bézier Curves (Postprint)

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Abstract

Compared with classical Bézier curves, Bézier curves with shape parameters provide degrees of freedom for shape adjustment independent of control points, yet simultaneously increase the workload for designers in selecting shape parameters. In view of this, the selection scheme for shape parameters is primarily discussed. First, it is proven that the parameter-extended basis of Bernstein basis functions given in existing literature constitutes a totally positive basis, thereby guaranteeing the theoretical value of the corresponding Bézier curves with shape parameters; then, the energy minimization method is employed to determine the values of shape parameters in the curve, and calculation formulas for the shape parameters are derived when the stretch energy, bending energy, and torsion energy of the curve are approximately minimized, providing convenience for the application of the curves.

Full Text

Energy Optimization of Bézier Curves with Shape Parameter

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Abstract: Compared with classical Bézier curves, Bézier curves with shape parameters provide shape adjustment freedom independent of control points, but simultaneously increase the workload for designers in selecting appropriate shape parameters. In view of this, this paper primarily discusses selection schemes for shape parameters. First, we prove that the parameter-extended basis functions of Bernstein basis functions given in existing literature constitute totally positive bases, thereby guaranteeing the theoretical value of the corresponding Bézier curves with shape parameters. Then, we employ an energy

minimization method to determine the shape parameter values in the curves, deriving calculation formulas for the shape parameter that approximately minimize the stretch energy, bending energy, and twist energy of the curves, thus providing convenience for practical applications.

Key Words: Bézier curve; shape parameter; energy optimization; parameter selection

0 Introduction

In the field of Computer-Aided Geometric Design (CAGD), curve and surface modeling is a fundamental and core problem. The Bézier method employs unique Bernstein polynomials as basis functions, endowing it with many excellent properties. Since its inception, it has received widespread attention from both industry and CAGD academia, becoming one of the most widely used basic methods and tools in curve and surface modeling. The Bézier method defines curves and surfaces through control points. Once the degree of the curve or surface is determined, its shape is uniquely defined by the control points. Therefore, to obtain a desired shape, the selection of control points is crucial. Reference [2] studied how to construct additional control points when some are given, such that the resulting Bézier curve has minimum energy.

Classical Bézier curves lack shape adjustment freedom independent of control points. Many references, such as [15-21], have addressed this issue by proposing new curve models with shape parameters that possess properties similar to Bézier curves and include Bézier curves as special cases. The advantage of such Bézier-like curve models with shape parameters lies in their ability to adjust curve shape by changing the shape parameter values without altering the control points. Reference [2] studied traditional Bézier curves, discussing how to compute unknown control vertices based on a selected energy optimization objective when some control vertices are known. However, there are cases where curve control vertices are predetermined, particularly when they are precise measurement points from physical objects, making them unsuitable for adjustment or computation. In such scenarios, the superiority of curve models with shape parameters becomes evident. To achieve the desired shape, one can select an appropriate energy function based on specific design requirements and then use optimization methods to determine the shape parameter values that minimize the energy function. This constitutes the research content of the present paper.

This paper focuses on the shape-parameterized curve model from reference [17], selecting three energy measures from reference [2] and deriving calculation formulas for the shape parameter that minimize these three energy measures.

1 Basis Function Representation

Reference [15] presented a function group consisting of three polynomial functions with parameter λ:

$$\begin{cases} b_0^2(t) = (1-t)^2 - \lambda(1-t) \\ b_1^2(t) = 2t(1-t) + \lambda, \quad t \in [0,1] \\ b_2^2(t) = t^2 - \lambda t \end{cases}$$

When $\lambda = 0$, the function group (1) becomes the quadratic Bernstein basis functions. Reference [17] extended the function group (1) using the recursive formula:

$$b_i^n(t) = (1-t)b_i^{n-1}(t) + tb_{i-1}^{n-1}(t), \quad i = 0, 1, \dots, n$$

When $\lambda = 0$, the function group (2) becomes the n-th degree Bernstein basis functions. Equations (1) and (2) together provide a parametric extension of the n-th degree Bernstein basis functions. For convenience, we will omit the argument notation when no confusion arises, such as abbreviating $b_i^n(t)$ as b_i^n.

Reference [20] presented an initial function group with two parameters α and β:

$$\begin{cases} b_0^2(t) = (1-t)^2 + \alpha t(1-t) \\ b_1^2(t) = 2t(1-t) \\ b_2^2(t) = t^2 + \beta t(1-t) \end{cases}$$

When $\alpha = \beta = -\lambda$, this function group is consistent with (1). Reference [20] also extended the initial function group using (2), and the function group $b_i^n(t)$ from reference [17] is a special case of this extension result. According to reference [20], when $\lambda \in (-2, 1]$, this function group is non-negative and linearly independent, thus forming a set of basis functions. For convenience, we refer to this function group as the n-th order B_λ-basis.

According to reference [20], the n-th order B_λ-basis can be uniformly and explicitly expressed as:

$$b_i^n(t) = \sum_{j=0}^{n} C_{i,j}^n(\lambda) B_j^n(t), \quad i = 0, 1, \dots, n$$

where $B_j^n(t)$ are the n-th degree Bernstein basis functions, and the coefficients $C_{i,j}^n(\lambda)$ are given by specific formulas. Based on (3), $b_i^n(t)$ can also be expressed as a linear combination of two $(n-1)$-th degree Bernstein basis functions:

$$b_i^n(t) = (1-t)b_i^{n-1}(t) + tb_{i-1}^{n-1}(t)$$

From (5), the relationship between n-th order B_λ-basis functions can be represented in matrix form as:

$$\mathbf{b}^n = \mathbf{J}_n \mathbf{b}^{n-1}$$

where $\mathbf{b}^n = (b_0^n, b_1^n, \dots, b_n^n)^T$, $\mathbf{b}^{n-1} = (b_0^{n-1}, b_1^{n-1}, \dots, b_{n-1}^{n-1})^T$, and $\mathbf{J}_n$ is a bidiagonal matrix of size $(n+1) \times n$.

2 Total Positivity of Basis Functions

References [17] and [20] did not discuss the total positivity of the B_λ-basis. We first present concepts and conclusions related to totally positive bases, then prove that the B_λ-basis is a totally positive basis.

Definition 1 (Totally Positive Matrix). A matrix is called totally positive if all its minors are non-negative.

Definition 2 (Totally Positive Basis). Let $\{u_0, u_1, \dots, u_n\}$ be a set of basis functions defined on a closed interval $[a, b]$. For any node sequence $t_0 < t_1 < \dots < t_m$ in $[a, b]$, if the collocation matrix:

$$M = \begin{pmatrix} u_0(t_0) & u_1(t_0) & \cdots & u_n(t_0) \\ u_0(t_1) & u_1(t_1) & \cdots & u_n(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ u_0(t_m) & u_1(t_m) & \cdots & u_n(t_m) \end{pmatrix}$$

is totally positive, then $\{u_0, u_1, \dots, u_n\}$ is called a totally positive basis.

Conclusion 1. The product of totally positive matrices is totally positive.

Conclusion 2. The classical Bernstein basis functions are totally positive.

Proposition 1. When $\lambda \in (-2, 1]$, for all $n \geq 2$, all elements of matrix $\mathbf{J}_n$ are non-negative.

Proof. We use mathematical induction. When $n = 2$, the matrix $\mathbf{J}_2$ is:

$$\mathbf{J}_2 = \begin{pmatrix} 1-\lambda & 0 \\ \lambda & 1+\lambda \\ 0 & 1-\lambda \end{pmatrix}$$

Under the condition $\lambda \in (-2, 1]$, all elements of this matrix are non-negative. Assume the proposition holds for $n = k$, i.e., for $\lambda \in (-2, 1]$, all coefficients on the right side of (5) are non-negative. Then for $n = k+1$, combining (2) and (5) we can derive:

$$b_i^{k+1}(t) = (1-t) b_i^k(t) + t b_{i-1}^k(t)$$

Under the induction hypothesis, all coefficients on the right side are non-negative for all $i = 0, 1, \dots, k+1$. This completes the proof.

Proposition 2. When $\lambda \in (-2, 1]$, for all $n \geq 2$, $\mathbf{J}_n$ is a totally positive matrix.

Proof. According to Proposition 1, all elements of $\mathbf{J}_n$ are non-negative. Since $\mathbf{J}_n$ is a bidiagonal matrix, it is easy to verify that all its minors are non-negative. Therefore, $\mathbf{J}_n$ is totally positive.

Proposition 3. When $\lambda \in (-2, 1]$, for all $n \geq 2$, the B_λ-basis is a totally positive basis.

Proof. For any given node sequence $t_0 < t_1 < \cdots < t_m$ in $[0, 1]$, let $\mathbf{B}^n$ and $\mathbf{b}^n$ denote the collocation matrices of the n-th degree Bernstein basis functions and the n-th order B_λ-basis, respectively. Then according to (6) we have:

$$\mathbf{b}^n = \mathbf{J}_n \mathbf{J}_{n-1} \cdots \mathbf{J}_2 \mathbf{B}^n$$

Since $\mathbf{J}_n, \mathbf{J}_{n-1}, \ldots, \mathbf{J}_2$ and $\mathbf{B}^n$ are all totally positive matrices, by Conclusion 1 we know that $\mathbf{b}^n$ is also totally positive. Hence, the B_λ-basis is a totally positive basis.

For convenience, we provide explicit expressions for equation (10) when $n = 2, 3, 4, 5$ and $k = 1, 2, 3$.

3 Selection of Shape Parameters

Given control vertices $\mathbf{P}_i \in \mathbb{R}^d$ $(i = 0, 1, \ldots, n)$, the B_λ-curve is defined as:

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i b_i^n(t), \quad t \in [0, 1]$$

From the properties of the B_λ-basis, the B_λ-curve possesses convex hull, symmetry, geometric invariance, affine invariance, and variation diminishing properties similar to Bézier curves. Additionally, since the B_λ-basis contains parameter λ, the B_λ-curve also has shape adjustability.

We select three energy functions used in reference [2]:

$$E_k = \int_0^1 \|\mathbf{P}^{(k)}(t)\|^2 dt, \quad k = 1, 2, 3$$

When $k = 1$, E_1 approximates stretch energy, reflecting curve length; when $k = 2$, E_2 approximates bending energy, reflecting curvature; when $k = 3$, E_3 approximates twist energy, reflecting the rate of curvature change.

We now derive the formula for solving parameter λ that minimizes E_k.

Substituting (4) into (7) and simplifying, we obtain:

$$E_k(\lambda) = \mathbf{P}^T \mathbf{D}_k(\lambda) \mathbf{P}$$

where $\mathbf{P} = (\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_n)^T$, and $\mathbf{D}_k(\lambda)$ is defined by (9).

To find the minimum, we take the derivative with respect to λ:

$$\frac{dE_k}{d\lambda} = \mathbf{P}^T \frac{d\mathbf{D}_k}{d\lambda} \mathbf{P} = 0$$

This yields:

$$\lambda = -\frac{\mathbf{P}^T \mathbf{X}_k \mathbf{P}}{\mathbf{P}^T \mathbf{Y}_k \mathbf{P}}$$

where $\mathbf{X}_k$ and $\mathbf{Y}_k$ are defined by (9).

For practical use, we provide explicit expressions for equation (10) when $n = 2, 3, 4, 5$ and $k = 1, 2, 3$.

4 Numerical Experiments

To visually demonstrate the modeling effect of determining shape parameters by the energy method, we present overall G^1 continuous B_λ-curves of degrees 2-5 (both open and closed) with appropriately chosen control vertices, along with corresponding Bézier curves of degrees 2–5 defined by the same control vertices.

First, consider an open 3rd-order B_λ-curve defined by 4 control vertices and a 3rd-degree Bézier curve. The red, green, and blue curves in Figure 1 [Figure 1: see original paper] correspond to $k = 1, 2, 3$ respectively, with parameters calculated from formula (10) as $\lambda = -0.8437$, $\lambda = -0.3548$, and $\lambda = -0.0995$. The black curve is the Bézier curve. Figure 1 shows that curves obtained with different energy optimization objectives exhibit different degrees of approximation to the control polygon; larger k values yield curves closer to the control polygon, while the $k = 3$ curve is very close to the Bézier curve.

[Figure 1: see original paper] shows 3rd-order B_λ-curve segments and 3rd-degree Bézier curve segments. The curvature plots for the red, green, and blue curves from Figure 1 are shown in subfigures 2(a)-(c) of Figure 2 [Figure 2: see original paper], while subfigure 2(d) shows the curvature plot for the black Bézier curve. Figure 2 reveals that the curve constructed with approximate minimum stretch energy has relatively large curvature, with maximum curvature at the endpoints; the curve with approximate minimum bending energy has relatively small curvature with the smallest curvature variation range; and the curve with approximate minimum twist energy exhibits zero curvature at endpoints, showing a curvature plot similar to that of Bézier curves.

Note: Although twist energy reflects the rate of curvature change, and E_3 approximates twist energy, Figure 2 shows that the curvature variation of the $k = 3$ B_λ-curve is not necessarily smaller than that of the $k = 2$ B_λ-curve. This discrepancy arises mainly because E_3 is an approximation rather than the exact energy value.

Figures 3 [Figure 3: see original paper]–10 [Figure 10: see original paper] show open and closed B_λ-curves of degrees 2–5 and their corresponding Bézier curves, with each curve segment' s parameter λ calculated according to (10). The starting segment of each B_λ-curve in these figures is marked with an arrow.

From Figures 3–10, we observe that curves obtained with approximate minimum stretch energy appear "angular," with each segment stretched straight, easily forming "sharp corners" at segment connection points—this characteristic becomes more pronounced with lower curve degrees. Curves obtained with approximate minimum bending energy consistently exhibit good visual effects. Curves obtained with approximate minimum twist energy mostly resemble the corresponding Bézier curves in shape. From the computed parameter data, their λ values are close to those of Bézier curves in some segments but differ significantly in others.

As mentioned earlier, when $\lambda \in (-2, 1]$, the B_λ-basis is non-negative and totally positive. The non-negativity of basis functions ensures the convex hull property of curves. However, since the parameter calculation formula derived from the energy optimization method is closely related to control vertex coordinates, it cannot guarantee that all obtained λ values fall within the interval $(-2, 1]$ in every case. Nevertheless, numerical examples show that this does not affect the convex hull property, because non-negativity of basis functions is a sufficient but not necessary condition for the convex hull property.

In practical applications, different curve segments can determine shape parameters according to different energy optimization objectives, or different energy optimization objectives can be weighted and combined to determine shape parameters. Designers can select the most suitable shape parameter determination scheme based on specific design requirements.

5 Conclusion

This paper studied Bézier curve models with shape parameters. First, we theoretically proved the application value of this model by demonstrating the total positivity property. Then, we derived calculation formulas for shape parameters that approximately minimize stretch energy, bending energy, and twist energy. Through curve plots and curvature plots, we intuitively compared the differences among various energy optimization objectives and presented modeling simulations of some common patterns in daily life.

Over the past fifteen years, Bézier curves with shape parameters have become a research hotspot in CAGD. In addition to references [15–21] cited in this paper, numerous other works have proposed new curves with properties similar to Bézier curves and containing shape parameters. Compared with existing literature, the advantages of this paper are mainly twofold: First, we examined the total positivity of basis functions, which determines the variation diminishing and convexity-preserving properties of corresponding curves, and thus determines their application value. In this sense, examining whether basis functions possess total positivity is one of the criteria for measuring the value of new curves. Second, we provided calculation formulas for shape parameter values based on energy optimization objectives. Introducing shape parameters into curves endows them with shape adjustment capability independent of control points, but also increases designers' workload in selecting shape parameters. Providing calculation formulas for shape parameters based on common design requirements helps designers quickly determine appropriate shape parameters according to specific design objectives, thereby facilitating practical applications.

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Note: Figure translations are in progress. See original paper for figures.