A multiresolution triangular plate-bending element method

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Abstract

First, based on the split-node shape functions of conventional triangular plate elements, basic full-node shape functions are constructed by extending the shape functions about the coordinate origin. These full-node shape functions are then scaled and translated across the element domain to generate a system of basis functions, which constitutes a simple and explicit mathematical foundation for multiresolution analysis. Simultaneously, this basis spans a sequence of mutually nested, hierarchically inclusive displacement subspaces, thereby establishing the concept of rational multiresolution analysis with resolution constants, and further constructing multiresolution quadrilateral plate-shell elements and their finite element method. Finally, through numerical examples, the following conclusions can be drawn: conventional triangular plate elements are single-resolution elements, representing a special case of the elements proposed in this paper; since the domain contains full nodes, the structural computational model established herein is a holistic model rather than a conventional mesh-fragmented model; the finite element method presented in this paper is a rational multiresolution analysis method, whose computational efficiency exceeds that of other irrational multiresolution analysis methods and can completely unify them; the clarity of structural numerical analysis is determined by the magnitude of resolution rather than mesh density; and the continuous full-node shape functions provide a theoretical basis for the conventional treatment method whereby nodal global stiffness can be obtained through manual superposition and assembly of element stiffnesses at common nodes.

Full Text

A Multiresolution Triangular Shell Element Method

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Abstract

This paper introduces the concepts of split nodes and full nodes based on the domain characteristics of traditional nodal shape functions. By extending the definition domain of split-node shape functions for conventional triangular shell elements using a parallelogram method about the coordinate origin, basic full-node shape functions are constructed. These full-node shape functions are then scaled and translated across the element domain to generate a basis function system that serves as a simple yet rigorous mathematical foundation for multiresolution analysis (MRA). This basis spans a displacement subspace sequence, thereby establishing the concept of rational multiresolution analysis with a resolution level constant. Building upon this foundation, a multiresolution triangular shell element and its computational method are further developed. Numerical examples validate the following conclusions: the resolution level defines an equally-spaced uniform distribution pattern of nodes within the domain; traditional triangular shell elements represent single-resolution elements and constitute a special case of the proposed element; because full nodes are employed throughout the domain, the structural computational model established herein is an integrated model rather than the conventional mesh-fragmented model; the proposed method is a rational multiresolution analysis approach that offers higher computational efficiency than other irrational multiresolution analysis methods while being capable of unifying them completely; the clarity of structural numerical analysis is determined by the resolution level rather than mesh density; and the continuity of full-node shape functions provides a theoretical basis for the conventional practice of assembling nodal stiffness through superposition of element stiffnesses at common nodes.

Keywords: triangular shell element; split node; full node; displacement subspace sequence; rational multiresolution analysis; resolution level (RL)

Introduction

The finite element method has become a universal approach for scientific research and engineering computations across various fields, profoundly impacting numerous industries. Shell elements represent a commonly used element type in finite element analysis for plate and shell structures prevalent in engineering practice. These elements are generally classified into rectangular and triangular configurations. Since rectangular elements are unsuitable for structures with complex geometric boundaries, triangular elements have found broader practical application. Triangular elements offer advantages such as efficient mesh generation through computational geometry algorithms like Delaunay triangulation and convenient local mesh refinement. However, despite these benefits, the absence of a systematic principle for predetermining mesh size, node count, and node placement makes it challenging even for experienced engineers to select appropriate mesh schemes for specific engineering problems, often requiring multiple iterations to achieve desired solution accuracy.

The fundamental difficulty in determining node count and placement for triangular meshes stems from inherent limitations in traditional finite element theory, manifesting in two primary aspects. First, the definition domain of triangular element shape functions is restricted to the element region, causing complete nodes (full nodes, shown as solid black circles in [Figure 2: see original paper]) to be fragmented into split nodes (shown as black portions in [Figure 1b: see original paper]). Consequently, mesh generation in conventional finite element analysis essentially discretizes the entire structure at element nodes. Second, uncertainty arises from the arbitrary nature of node distribution—nodes may be arranged in regular uniform patterns or random irregular configurations—due to the lack of a governing principle for controlling node quantity and distribution, i.e., the absence of a robust mathematical foundation. Generally, multiresolution analysis refers to techniques where the number of information details (nodes) can be altered arbitrarily. Multiresolution analysis that modifies node count through uniform distribution (supported by a robust mathematical basis) is termed rational multiresolution analysis, whereas approaches employing random, irregular distributions (lacking mathematical support) are called irrational multiresolution analysis. Evidently, the fragmentation of complete information details (full nodes) and the reliance on irrational multiresolution analysis represent two inherent deficiencies in conventional finite element theory.

Over the past three decades, researchers worldwide have proposed various innovative methods to eliminate mesh-related problems. Professor Belytschko et al. introduced the element-free Galerkin method in 1994, which constructs full-node shape functions for target points by determining polynomial coefficients based on conditions satisfied by surrounding points. In 1995, Professor Ko et al. developed the wavelet finite element method by constructing domain-wide full-node shape functions using the tensor product of Daubechies wavelet scaling functions. Professor Hughes et al. established isogeometric analysis in 2005, which builds full-node shape functions through tensor products of spline functions. These methods share a common characteristic: they construct full-node shape functions within the domain to create integrated structural models rather than the discretized models requiring mesh division in traditional finite element analysis. Consequently, these approaches eliminate the need for mesh generation and demonstrate advantages for certain specialized engineering problems. However, because node placement in these methods remains somewhat arbitrary and lacks mathematical foundation, their computations still fall under the category of irrational multiresolution analysis.

To address the arbitrary node placement issue and the lack of mathematical foundation, this study proposes a multiresolution element method. This approach extends split-node shape functions to create basic full-node shape functions, which are then scaled and translated across the element domain to construct a basis function system—a concise yet rigorous mathematical foundation. This basis spans a displacement subspace sequence, forming a novel multiresolution analysis framework characterized by a resolution level constant (RL). The resolution level defines a principle for uniform node distribution, enabling free adjustment of node count by modifying the element resolution constant. Once the resolution constant is determined, both node quantity and location are specified, thereby resolving the uncertainty issues regarding node count and placement inherent in traditional finite element and other methods. Resolution refers to the minimum distance at which differences between two information details (physical quantities) can be distinguished. Uniform resolution therefore requires identical spacing between adjacent nodes throughout the domain. The multiresolution element method thus represents a rational multiresolution analysis approach supported by a robust mathematical foundation, operating without meshes and employing resolution constants to systematically determine node count and placement. This paper derives the computational formulation for multiresolution analysis of triangular thin shell elements, providing a powerful alternative tool for analyzing plate and shell structures in engineering practice.

1 Basic Full-Node Shape Functions

Consider a triangular thin shell element with the rectangular coordinate system shown in [Figure 1a: see original paper] and the mid-surface coordinate system depicted in [Figure 1b: see original paper]. The dimensionless equations for the three edges of the triangle can be expressed as:

For the base edge 12:
For the side edge 23:
where a and b represent intercepts on the horizontal and vertical axes, respectively.
For the side edge 13:

The displacement at any point on the triangular shell element can then be written as:

where u, v, and w are displacements along the x, y, and z coordinate axes, respectively; u_i, v_i, and w_i are displacements at element node i ($i = 1, 2, 3$) along the respective axes; θ_{xi} and θ_{yi} are rotations about the x and y axes at node i; N_i^m represents the in-plane membrane displacement shape function at node i; N_i^b is the bending shape function at node i; and N_{xi}^b, N_{yi}^b are rotation shape functions about the x and y axes at node i.

The shape functions in the above equation can be expressed as [1]:

where L_1, L_2, and L_3 are the area coordinates of the three nodes of the triangular element.

From equations (1), (2), and (3), the following relationships can be obtained:

with $(i = 1, 2, 3)$, and within the triangular element domain D_1, We can now construct full-node shape functions using the split-node shape functions for triangular shell elements. As shown in [Figure 1b: see original paper], the nodes of a triangular element are fragmented by the element domain (shaded area), meaning the element splits its nodes. Consequently, mesh generation in traditional finite element analysis discretizes the entire structure at each node. To establish a non-discrete integrated structural model, complete nodes—full-node shape functions—must be constructed. The key to constructing these functions lies in extending the definition domain of nodal shape functions to enclose the node. For triangular elements, this can be achieved by creating a series of parallelograms around the origin (node) based on the original triangle, effectively superimposing identical triangles to expand the node' s definition domain. The result is a hexagon as shown in [Figure 2: see original paper], where the node at the origin is enclosed by its definition domain (shaded area), making it a full node. The shape function for this basic full node is presented below.

The basic full-node shape function is formed by superimposing six triangular split-node domains. The resulting function is continuous and possesses the Kronecker delta property:

2 Displacement Subspace Sequence

To enable multiresolution computational analysis, we must construct a multiresolution displacement subspace sequence for the element. Through scaling and translation of functions, a basis function system for the subspace sequence can be established. By scaling and translating the basic full-node shape function within the domain, the basis function vector Ψ_e^n for the element' s multiresolution displacement subspace sequence is formed. For a triangular thin shell element with resolution level RL, Ψ_e^n can be expressed as:

where n is a positive integer representing the scaling factor for each side of the triangle; $r,\ s = 0,\ 1,\ 2,\ 3,\ \cdots$are parameters representing node position translations; and $(x, y) \in D$.

First, the basic full-node shape function is scaled within the domain, then the scaled shape function is translated to all nodes in the element domain, thereby constructing shape functions for each full node within the element region.

As illustrated in [Figure 4: see original paper], shape functions for full nodes within the triangular element domain can be obtained through scaling and translation of the basic full-node shape function. Similarly, shape functions for boundary nodes (B) can be derived, but with the portion extending beyond the triangular element domain by 1/2 removed, making boundary nodes half-nodes. For the three vertex nodes (T) of the triangular element, shape functions are also obtained through scaling and translation, but with five of the six hexagonal portions extending beyond the domain removed, making the vertex nodes split nodes.

[Figure 4: see original paper] shows that nodes within the element are uniformly distributed. Since the basis function vector contains elements φ_{nrs}, ϕ_{xnrs}, and ϕ_{ynrs}, different combinations of n values and node translation parameters form linearly independent function groups. Consequently, the resulting displacement space sequence comprises multiple subspaces, constituting a multiresolution analysis (MRA) framework: $V_0 \subset V_1 \subset V_2 \subset \cdots$. When $I = 2^i$, we have:

where $\mathbb{Z}$ is the set of positive integers and V_i represents the displacement subspace at resolution level $\mathrm{RL} = (2^i + 1)$.

Therefore, within the displacement subspace at resolution level $\mathrm{RL} = (2^i + 1)$, the displacement at any point in the shell element can be expressed as:

where u_{rs}, v_{rs}, w_{rs}, θ_{xrs}, and θ_{yrs} are the displacements along the x, y, z directions and rotations about the x, y axes at node (r, s), respectively.

This formulation demonstrates that nodes within the element are regularly and uniformly distributed, with their quantity and location freely adjustable through the resolution constant. Once the element resolution constant is determined, the total number of nodes and their specific positions are defined. When the resolution level $\mathrm{RL} = 1 \times 3$ $(n = 1)$ in equation (13), it reduces to the displacement field of a conventional triangular shell element as shown in equation (4). Thus, traditional triangular shell elements are single-resolution elements, representing a special case of the proposed multiresolution triangular shell element.

3 Multiresolution Triangular Shell Element

In the displacement subspace with resolution level $\mathrm{RL} = (2^i + 1)$, the total potential energy of a triangular shell element can be expressed as:

where p_x, p_y are distributed forces along the x and y axes; q is the transverse distributed force; P_{ix}, P_{iy} are concentrated forces along the x and y axes at specific element locations; and Q_i is a transverse concentrated load at a specific element location. Additionally:

with E representing the material' s elastic modulus, t the shell element thickness, and μ Poisson' s ratio.

Substituting equations (15) and (13) into equation (14) and rearranging yields:

where K_e^n is the element stiffness matrix, f_e^n the element nodal distributed equivalent load vector, and F_e^n the element nodal concentrated equivalent load vector.

Applying the principle of minimum potential energy, $\frac{\partial \Pi}{\partial a} = 0$, yields the equilibrium equation for the shell element:

where k_{cdrs} represents the stiffness matrix correlating nodes (c, d) and (r, s), which can be expressed as:

with N being the shape function matrix and P the concentrated load vector.

4 Transformation Matrix

The preceding steps yield the stiffness matrix and equivalent nodal forces for the multiresolution triangular shell element. For structural analysis, the element nodal directions in the local coordinate system must be transformed to the global coordinate system. This transformation of element stiffness and equivalent nodal forces to the global system can be accomplished through the transformation matrix T_e^n:

where θ is the angle between the element local coordinate axes and the global coordinate axes.

5 Structural Global Stiffness and Nodal Forces

After transforming to the global coordinate system, the element boundary split nodes are obtained. Superimposing the corresponding split nodes on common boundaries of adjacent elements (through domain extension) readily yields the structural global stiffness matrix and nodal force vector.

6 Numerical Examples

Example 1 [7]. Consider a parallelogram skew plate with simply supported opposite edges, side length L, and skew angle of 60°, as shown in [Figure 5: see original paper]. With Poisson' s ratio $\mu = 0.3$ and subjected to uniform load q, determine the deflection at the plate center.

For the central deflection of the skew plate, we employ the proposed multiresolution method, conventional single-resolution triangular shell element FEM, and the interval cubic B-spline wavelet FEM (BSWI). Due to the use of full-node shape functions, the multiresolution triangular shell element model is an integrated structural model requiring only two elements (and) to represent the entire skew plate. The resolution constants for each element are taken as 5×9, 7×13, and 9×17, corresponding to resolutions of $L/8$, $L/12$, and $L/16$, respectively. The resulting resolution constants for the entire skew plate are 9×9, 13×13, and 17×17, with plate resolutions matching the element resolutions at $L/8$, $L/12$, and $L/16$, as shown in [Figure 6a: see original paper]. According to the resolution definition, nodes must be uniformly distributed within elements and equally spaced throughout the structure. Consequently, once the resolution constant is determined, node count and location are fixed. In contrast, conventional FEM employs split-node shape functions, creating a discretized model with triangular meshes of 8×8, 12×12, and 16×16, as shown in [Figure 7b: see original paper]. Clearly, this mesh node distribution lacks uniformity requirements, and node positions can vary. The interval cubic B-spline wavelet FEM also uses an integrated structural model constructed from full-node shape functions built via tensor products of spline scaling functions.

The computational results are listed in . The table demonstrates that computational clarity can be improved by adjusting the multiresolution element' s resolution. The element functions like an image where nodes correspond to pixels, while conventional single-resolution elements improve accuracy through mesh refinement. Although accuracy and clarity are comparable, the multiresolution element method offers several advantages: (1) The resolution in multiresolution models has clear mathematical meaning and a solid mathematical foundation, whereas traditional mesh generation relies primarily on experience without mathematical basis; (2) The integrated multiresolution model contains only two elements, resulting in fewer vector transformations (one per node) compared to the numerous transformations in conventional discretized models (six per node), yielding higher computational efficiency. Compared with BSWI results, the multiresolution element method achieves comparable clarity but with distinct advantages: (1) The resolution has explicit mathematical meaning and foundation, while B-spline node count selection relies on experience; (2) Full-node shape functions have simple expressions with Kronecker delta properties, facilitating boundary condition treatment and improving efficiency, whereas tensor-product full-node shape functions are complex, may have negative values, complicate numerical integration, and lack Kronecker delta properties, requiring special boundary condition treatment that significantly reduces computational speed.

Example 2 [8]. Consider a circular thin plate of radius r and thickness t, simply supported or clamped along its periphery, subjected to transverse uniform pressure q. The material has elastic modulus E and Poisson' s ratio $\mu = 0.3$. Determine the displacement and bending moment at the plate center.

Due to structural and loading symmetry, a quarter circular plate can be used for analysis. Both multiresolution integrated and conventional single-resolution discretized models are employed, as shown in [Figure 7: see original paper]. The multiresolution integrated model consists of two triangular elements (and) with resolution constants of 2×5 and 4×7, while the conventional FEM uses meshes of 3×6 and 7×13. Results for clamped (SS) and simply supported (SC) conditions are presented in . The results show that the multiresolution triangular shell element method can enhance computational clarity by adjusting resolution—more uniformly distributed nodes yield higher clarity. The two elements can be assembled at interface split nodes to form full nodes, completing the structural analysis. Compared with conventional FEM, the multiresolution method uses significantly fewer elements, drastically reducing vector transformations and substantially improving computational speed. Moreover, with mathematical foundation support, using resolution to determine node count, placement, and accuracy description makes the multiresolution approach more scientific and efficient than traditional mesh-based methods.

Example 3 [9]. Consider a folded plate structure shown in [Figure 8: see original paper] with elastic modulus $E = 2.06 \times 10^5$ MPa, Poisson' s ratio $\mu = 0.3$, thickness $t = 1$ m, square plan dimensions $L = 50$ m, subjected to uniform external pressure $q = 1$ kN/m². Boundaries AB and EF are free, while all other edges are simply supported. Determine displacements along edges AB and CD.

This problem is solved using both the proposed multiresolution triangular shell element method and conventional single-resolution triangular shell element FEM. To satisfy the resolution requirement of uniform node distribution within elements and equal spacing throughout the structure, midpoints C and D are taken on the two base edges, connecting them to midpoint O on the ridge line. This divides the folded plate into four equal square regions, each further subdivided into triangles. The resulting multiresolution integrated model comprises eight multiresolution triangular shell elements (through), as shown in [Figure 9a: see original paper], with each element having a resolution constant of 3×7 ($n = 5$) and the entire folded plate structure having a resolution constant of 11×11. The uniformly distributed nodes yield a resolution of 5 m. Since the domain contains full nodes, this constitutes a meshless integrated model, with results corresponding to case A curves in [Figure 10: see original paper]. The single-resolution model consists of 10×10 conventional triangular shell elements ([Figure 9b: see original paper]), representing a mesh-discretized model with split nodes, with results corresponding to case B curves. A third curve in [Figure 10: see original paper] shows results from [9] using interval cubic B-spline wavelet FEM, which also forms a meshless integrated model but employs tensor-product full-node shape functions lacking Kronecker delta properties, requiring transformation matrices for boundary conditions that significantly reduce computational efficiency.

[Figure 10: see original paper] demonstrates that the multiresolution triangular shell element method combines advantages of both FEM and meshless methods—it allows domain subdivision based on structural geometry and loading characteristics while enabling adjustment of node count through resolution modification to control computational clarity. Although clarity is comparable to conventional single-resolution models, the multiresolution approach offers distinct advantages: (1) Resolution has explicit mathematical meaning and foundation, unlike conventional mesh generation based on experience; (2) Since full nodes are employed, element nodal stiffness and equivalent forces can be generated automatically, whereas conventional models require manual superposition at split nodes; (3) The integrated multiresolution model uses fewer elements than fragmented conventional models, resulting in fewer vector transformations and higher computational efficiency.

Conclusions

This paper presents a multiresolution triangular shell element method with the following characteristics compared to conventional triangular shell element FEM:

1) The concepts of split nodes and full nodes for triangular shell elements are introduced. Based on conventional split-node shape functions, a domain extension method is proposed to construct basic full-node shape

functions with Kronecker delta properties, facilitating boundary condition treatment. The continuity of these functions provides theoretical justification for the conventional practice of assembling nodal stiffness through superposition of element stiffnesses at common nodes.

2) Scaling and translation of basic full-node functions construct basis functions—a concise and rigorous mathematical foundation for multiresolution analysis—forming a displacement subspace sequence for multiresolution elements. The resolution level constant (RL) is introduced, establishing the concept of rational multiresolution analysis and enabling rationalized multiresolution numerical analysis of structures.

3) Conventional triangular shell elements are single-resolution elements, representing a special case of the proposed multiresolution triangular shell element.

4) Resolution defines equally-spaced uniform node distribution within the domain. Structural analysis clarity is determined by resolution magnitude rather than mesh density; higher-clarity elements contain more nodes than lower-clarity elements.

5) The proposed multiresolution triangular shell element method achieves higher computational efficiency than other meshed or meshless shell element FEMs. It replaces conventional accuracy with clarity, mesh/elements with resolution, irrational multiresolution analysis with rational analysis, and split nodes with full nodes, thereby transforming structural computational models from fragmented (discretized) to integrated formulations.

6) With the introduction of multiresolution element methods [10,11,12,13], rational multiresolution analysis will find extensive application in engineering practice.

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Note: Figure translations are in progress. See original paper for figures.