Derivation Modules of Cycle Graph Hyperplane Arrangements and Their Free Resolutions

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Abstract

This paper employs relevant theories from algebra and combinatorics to investigate the freeness of hyperplane arrangement theory, presents explicit forms for the generators and derivation modules of graphic arrangements corresponding to cycle graphs, and provides a combinatorial interpretation of the derivation modules and their free resolutions for cycle graph arrangements in graph-theoretic terms.

Full Text

Derivation Modules of Cycle Graph Hyperplane Arrangements and Their Free Resolutions

Wu Jun

Abstract: This paper uses relevant theories from algebra and combinatorics to carry out a study of the freeness of hyperplane arrangements. It gives the generators of the graph arrangement corresponding to a cycle graph and the concrete form of its derivation module, and provides a corresponding combinatorial interpretation, in graph-theoretic form, of the derivation module of a cycle graph arrangement and its free resolution.

Keywords: hyperplane arrangement; derivation module; cycle graph; free resolution

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I. Introduction

In algebraic geometry, the singularity problem has always been one of the important objects of study. After localizing an isolated singularity of an algebraic cluster, all tangent hyperplanes passing through the isolated singularity constitute a hyperplane arrangement. On the one hand, one can study algebraic clusters through hyperplane arrangements; on the other hand, because the combinatorial objects of hyperplane arrangements—the geometric lattice, or the equivalent object, the simple matroid—belong to topics long of concern to combinatorialists and lattice theorists, the study of hyperplane arrangements naturally includes geometry, algebra, topology, and combinatorics, among other aspects [1, 2].

In 1970, in order to study the Gauss-Manin connection of isolated singularities, K. Saito introduced the concept of a free divisor [3]. Almost at the same time, the concept of a free arrangement of hyperplanes also began to be established by H. Terao. Later, Silvotti and Schenck [4-6] and others pointed out that the freeness problem for hyperplane arrangements is equivalent to the problem of whether, in projective space, a reflexive sheaf splits into a direct sum of line bundles—that is, the splitting problem. Around the same period, H. Terao proposed a still-unresolved···

the well-known conjecture it addresses: the freeness of a hyperplane arrangement is determined combinatorially [1,2]. This conjecture has become a central problem in the current study of freeness of hyperplane arrangements. This paper also investigates the freeness of arrangements determined by a class of combinatorial objects, namely cycle graphs.

II. Intersections and Derivation Modules of Hyperplane Arrangements

For an l-dimensional real vector space $V = R^l$, let $\{x_1, x_2, \cdots, x_l\}$ be a set of bases. For $V^* = V \setminus \{\text{origin } 0\}$, let $S = S(V^*)$ be the graded symmetric algebra on V^*, that is,

$$S = \bigoplus_{k \geq 0} S_k,$$

where S_k is the submodule of degree k, and $R = S_0$. For any homogeneous (homogeneous of degree one) polynomial $\alpha \in S_1$, $H = \ker(\alpha)$ is called a hyperplane; that is, H is an $l-1$-dimensional subspace passing through the origin. We also write $\alpha = \alpha_H$. A **hyperplane arrangement** (or simply an **arrangement**) is the collection of a finite number of hyperplanes,

$$A = \{\ker(\alpha_H) \mid \alpha_H \in S_1\}.$$

For an arrangement A, all possible intersections of hyperplanes in A form, under the reverse inclusion relation among subspaces, a partially ordered set $L = L(A)$. In L, we define the join and meet operations of any two hyperplanes H_i and H_j in A, respectively, by

$$H_i \vee H_j = H_i \cap H_j, \quad H_i \wedge H_j = span\{H_i, H_j\},$$

and V and $\bigcap_{H \in A} H$ are respectively the minimum element and maximum element of L. Thus L forms a geometric lattice, called the intersection lattice of the arrangement. The intersection lattice is a combinatorial object in the theory of hyperplane arrangements. On the one hand, this object is very closely related to the geometry, algebra, and topology of the arrangement; on the other hand, geometric lattices are equivalent to simple matroids. The study of matroids is a broad and interesting field, and the fact that Terao' s conjecture remains difficult to resolve is also related to our insufficient understanding of geometric lattices.

continues to attract the unremitting study of related scholars [7-10].

At the same time, for the polynomial ring S, we introduce the first-order derivations

$$D_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, l$$

satisfying the Newton–Leibniz formula $D_i(fg) = gD_i(f) + fD_i(g), (\forall f, g \in S)$. Thus one obtains an S-graded S-module (derivation module)

$$D = \bigoplus_{i=1}^{l} SD_i.$$

Combining this derivation module with the arrangement A as follows,

$$D(A) = \{\theta \in D \mid \theta(\alpha_H) \in S\alpha_H,\ H \in A\},$$

the resulting S-graded submodule $D(A)$ is called the derivation module of the arrangement A. If we write

$$Q(A) = \prod_{H \in A} \alpha_H,$$

then it is well known that $\theta \in D(A)$ is equivalent to $\theta(Q(A)) \in S \cdot Q(A)$. If $D(A)$ is a free (graded) S-module, then A is called a free arrangement. Reflection arrangements introduced from the theory of root systems of finite reflection groups and certain deformations of them, as well as supersolvable arrangements, are all examples of free arrangements. The derivation modules of free arrangements are also closely related to certain Chen classes [3–6].

In general, the derivation modules of most arrangements are not free modules, and finding a free resolution of a derivation module becomes a natural approach to studying the properties of arrangement derivation modules. Regardless of whether Terao' s conjecture is correct, it suggests that when studying free resolutions of derivation modules we should also relate them to the combinatorial objects of arrangements. This paper uses precisely this idea to study free resolutions of derivation modules of cycle graph arrangements.

III. Derivation Modules of Cycle Graph Arrangements and Their Free Resolutions

Let the vertex set of a (simple) graph G be $\{x_1, x_2, \ldots, x_l\}$. If its directed edge set is

$$E(G) = \{e_{i,i+1} \mid x_i \text{ is connected to } x_{i+1},\ i = 1, 2, \ldots, l-1\} \cup \{e_{l,1} \mid x_l \text{ is connected to } x_1\},$$

then the graph G is called a cycle graph. It is easy to see that in this case $E(G)$ has altogether l edges. Denote the corresponding edges by

the set of linear polynomials is $\{\alpha_i = x_i - x_j \mid e_{i,j} \in E(G)\}$. Define the graphical arrangement of G to be

$$A(G) = \{\ker(\alpha_i) \mid i = 1, 2, \cdots, l\},$$

then the arrangement $A(G)$ contains l hyperplanes and

$$\dim\left(\bigcap_{H\in A(G)} H\right) = 1.$$

Note that the normal vectors of any $l-1$ hyperplanes in this arrangement are linearly independent, and the normal vectors of the l hyperplanes form the unique linearly dependent set. Hence its intersection lattice is the upper 1-truncation of the Boolean algebra on l elements.

Denote the derivation module of the arrangement $A = A(G)$ by $D(G)$, and let $Q = Q(A(G))$. If

$$\theta_{ij} = \alpha_i\alpha_j[(D_i - D_{i+1}) - (D_j - D_{j+1})],$$

where $1 \le i < j \le l$, then we have the following conclusion.

Theorem 1. The minimal generating set of the derivation module $D(G)$ of the cycle graph G is

$$\{\theta_0, \theta_E\} \cup \{\theta_{ij} \mid 1 \le i < j \le l\},$$

where $\theta_0 = D_1 + D_2 + \cdots + D_l$, and

$$\theta_E = x_1D_1 + x_2D_2 + \cdots + x_lD_l$$

is the Euler derivation.

Proof: Let $Q = \alpha_1\alpha_2\cdots\alpha_l$ and

$$\theta = f_1D_1 + f_2D_2 + \cdots + f_lD_l \in D(G),$$

where $f_i \in S_k,\ 1 \le i \le l$, that is, f_i is a homogeneous polynomial of degree k, and k is a nonnegative integer. Since

$$\theta(Q) \in SQ,$$

i.e. $Q \mid \theta(Q)$, and

$$\theta(Q) = Q\cdot\left[\frac{1}{\alpha_1}(f_2 - f_1) + \frac{1}{\alpha_2}(f_3 - f_2) + \cdots + \frac{1}{\alpha_{l-1}}(f_{l-1} - f_{l-2}) + \frac{1}{\alpha_l}(f_1 - f_l)\right],$$

we obtain

$$g = \frac{1}{\alpha_1}(f_2 - f_1) + \frac{1}{\alpha_2}(f_3 - f_2) + \cdots + \frac{1}{\alpha_{l-1}}(f_l - f_{l-1}) + \frac{1}{\alpha_l}(f_1 - f_l) \in S.$$

Thus

1) When $k = 0$, we have $g = 0$, and may take $f_1 = f_2 = \cdots = f_l = C$, where C is a nonzero constant. Clearly in this case $\theta \in S\theta_0$.

2) When $k = 1$, we have $g = C$, where C is a nonzero constant, and may take $f_i = -Cx_i$,

$1 \leq i \leq l$. Clearly, at this time $\theta \in S\theta_E$.

3) When $k = 2$, there is $g = C_1x_1 + \cdots + C_lx_l \in S_1$ (where $C_1, C_2, \cdots, C_l$ are constants not all zero). Note that for any i ($1 \leq i \leq l$), if $\alpha_{i-1} \mid (f_i - f_{i-1})$ and $\alpha_i \mid (f_{i+1} - f_i)$, then one may set $f_j = \sigma_{ij}C_i'\alpha_{i-1}\alpha_i$, where $1 \leq j \leq l$, σ_{ij} is the Kronecker symbol, and C_j' is uniquely determined by the expression of g. Then, by the arbitrariness of $C_1, C_2, \cdots, C_l$ and i, it follows that at this time θ must belong to the S-module generated by $\{\theta_{ij} \mid 1 \leq i < j \leq l\}$; moreover, when $k \geq 2$, θ can be generated by $\{\theta_0, \theta_E\} \cup \{\theta_{ij} \mid 1 \leq i < j \leq l\}$. Note that, after quotienting out the θ_0 part of the submodule generated by $\theta_{ij}, 1 \leq i < j \leq l$, one obtains that $\theta_{ij}, 1 < i < j \leq l$ are minimal generators of the quotient module.

In summary, we know that

$$D(G) = S\theta_0 \oplus S\theta_E \oplus \left(\sum_{1<i<j\leq l} S\theta_{ij}\right),$$

and clearly $\{\theta_0, \theta_E\} \cup \{\theta_{ij} \mid 1 < i < j \leq l\}$ is a minimal generating set, with a total of $\binom{l-1}{2} + 2$ elements. (Proof completed.)

IV. Free Resolution of the Derivation Module

Among the generating sets of $D(G)$, θ_E is a degree-one generator that appears for every arrangement, while the appearance of the degree-zero generator θ_0 is precisely because

$$\dim\left(\bigcap_{H\in A(G)} H\right) = 1.$$

At this time, $D(G)$ is not a free S-module. We begin to discuss its graded free resolution as an S-module.

Definition 1. For an arrangement $A(G)$ of a graph G, if there is a sequence of graphs

$$G \xrightarrow{i_1} G_1 \xrightarrow{i_2} G_2 \xrightarrow{i_3} \cdots \xrightarrow{i_n} G_n,$$

where G_k is obtained by adding an edge to G_{k-1}, and i is the natural inclusion relation, and if the corresponding sequence of derivation modules of these graph arrangements is as follows:

$$D(G_n) \xrightarrow{i_n^*} \cdots \xrightarrow{i_3^*} D(G_2) \xrightarrow{i_2^*} D(G_1) \xrightarrow{i_1^*} D(G)$$

if the $D(G_n)$ in the sequence is a free module, where $i_\bullet^*$ is the module homomorphism naturally induced by $i_\bullet$, then the sequence in the above diagram is called a free resolution of G; the smallest n among all free resolutions is called the **depth** of the free resolution of G.

To study free resolutions of cycle graphs, we also need to introduce some knowledge concerning graph arrangements.

Definition 2. A graph G is called a **chordal graph** if every cycle of G is generated by triangles.

Theorem 2 [11]. The arrangement of a graph G is supersolvable if and only if the graph G is chordal.

Theorem 3 [11]. The arrangement of a graph G is free if and only if G is supersolvable.

From Theorems 2 and 3 it follows that Terao' s conjecture holds for graph arrangements. At the same time, it is also known that, if one wants to find a free resolution of a cycle graph G, this is equivalent to finding a minimal chordal graph containing G. We have the following conclusion.

Theorem 4. The depth of a free resolution of a cycle graph G with l vertices is $l-2$.

Proof. Suppose the vertex set of G is $\{x_1, x_2, \cdots, x_l\}$, and the edge set is obtained by connecting these vertices successively from head to tail. Connect the vertex x_1 successively with the $l-2$ vertices $x_2, x_3, \cdots, x_{l-1}$ that are not adjacent to it, thereby obtaining in succession a sequence of graphs consisting of $l-2$ graphs,

$$G \xrightarrow{i_1} G_1 \xrightarrow{i_2} G_2 \xrightarrow{i_3} \cdots \xrightarrow{i_{l-2}} G_{l-2},$$

It is easy to verify that G_{l-2} is a chordal graph, and because deleting any one of the $l-2$ edges added to G_{l-2} makes it no longer chordal, $l-2$ is already minimal; hence it is precisely the depth of a free resolution of the cycle graph G. (Proof completed.)

For the derivation modules of the $l-2$ graph arrangements in the free resolution in the proof of Theorem 4, the corresponding generators and induced module homomorphisms in the following sequence of derivation modules all exist,

$$D(G_{l-2}) \xrightarrow{i^*_{l-2}} \cdots \xrightarrow{i^*_3} D(G_2) \xrightarrow{i^*_2} D(G_1) \xrightarrow{i^*_1} D(G).$$

These modules and their maps can be listed by means of the theory of algebraic spectral sequences. In combinatorial terms, this may be interpreted as follows: each triangle of G_{l-2} corresponds to a basis element of $D(G_{l-2})$, and the action of this basis element on the two edges among the three corresponding edges that contain the vertex x_1 is nonzero, while its action on all other edges of G_{l-2} is zero. The situation in the above sequence of derivation modules is similar. Combinatorially, the process of successively adding edges to the cycle graph is reflected algebraically in the gradual elimination of derivations in the derivation module that do not satisfy this condition. For simplicity, here we give only the free module

$$D(G_{l-2}) = S\theta_0 \oplus S\theta_E \oplus D',$$

where the free submodule

$$D' = \bigoplus_{j=2}^{l-1} S\theta_{1j}.$$

For a general arrangement, the situation is much more complicated. Some earlier progress was made by Yuzvinsky [12], and subsequently by Yoshinaga [13], and more recently by Abe [9], among others.

Remark: The cycle graphs considered in this paper are graphs with relatively simple structure. For their corresponding arrangements in the complex space, the combinatorial and topological objects, such as the characteristic polynomial (equivalently, the Poincaré polynomial of its complement) and the fundamental group, can all be given combinatorial interpretations. Studies of this type, similar to Theorem 4, will also provide inspiration and ideas for general arrangements, and are also one of the common approaches currently used in the search for new free arrangements.

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The Derivation Module and Its Free Resolution of Circuit Graph Arrangements

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Abstract: We focus on a circuit graph arrangement and its free resolution from algebraic and combinatorial perspectives. We detected the generators of the derivation module and find out a basis for a free resolution. At last, we offer a combinatorial expression of the free resolution.

Keywords: arrangement of hyperplanes; derivation module; circuit graph; free resolution

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