Power Law of Shear Viscosity in Einstein-Maxwell-Dilaton-Axion Model (Postprint)

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Full Text

Preamble

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Abstract

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Keywords: breaking of translational invariance, shear viscosity, gauge/gravity duality, hyperscaling violation, Einstein-Maxwell-Dilaton-Axion model

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1 Introduction

Over the last decade, the development of holographic techniques has provided new insights into hydrodynamics with strong coupling. One celebrated achievement is the Kovtun-Son-Starinets (KSS) bound for the ratio of shear viscosity to entropy density [?], which is considered a fundamental bound for near-perfect fluids with strong interactions. However, in recent years counter-examples that violate the KSS bound have been found in holographic literature, including higher-derivative gravity [?, ?], anisotropic systems [?, ?, ?, ?], as well as isotropic systems without translational invariance [?, ?, ?, ?, ?, ?, ?, ?]. In the latter case, shear viscosity loses its hydrodynamical interpretation because of momentum non-conservation and is usually defined by the Kubo formula:

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G_R^{xy,xy}(\omega, k=0)$$

where $G_R^{xy,xy}$ is the retarded Green' s function of the energy-momentum tensor operator $\hat{T}^{xy}$ in the dual boundary theory. Nevertheless, it remains quite instructive to investigate the temperature behavior of the ratio of shear viscosity to entropy density in general holographic models without translational invariance.

Historically, the breaking of translational invariance was introduced in holography to study the transport properties of dual systems with momentum dissipation [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. In this setup, new geometries may emerge in the IR, often accompanied by new scaling relations that lead to novel scaling behaviors of thermodynamic quantities or Green' s functions with temperature T or frequency ω [?, ?, ?, ?, ?, ?, ?, ?, ?].

In particular, through recent progress in [?], it has been learned that when translation symmetry breaking is relevant in the far IR, the ratio exhibits a power-law behavior with temperature:

$$\frac{\eta}{s} \sim T^{\kappa}$$

which reflects the scaling symmetry emerging in the IR. Moreover, a new bound for the exponent κ was proposed as $\kappa \leq 2$ in [?], which might be supported by a heuristic argument based on the bound for the rate of entropy production:

$$\frac{d\log(s)}{d\log(T)} \leq 1$$

Other holographic models investigated in [?, ?, ?, ?, ?, ?] also satisfy the bound $\kappa \leq 2$. However, it was soon found in [?] that $\kappa > 2$ is possible when Lifshitz scaling [?, ?, ?] or hyperscaling violation [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] emerges in the IR. While the bound for the rate of entropy production is not violated, since the origin of the power law should be understood as the nontrivial anomalous dimensions under the rescaling of the IR solution [?, ?].

Furthermore, in [?] we analytically derived a formula for κ in a general Einstein-Maxwell-Dilaton-Axion (EMD-Axion) model with spatial dimension d, dynamical critical exponent z, and hyperscaling violating exponent θ. Specifically, we found:

$$\kappa = \frac{d+z-\theta}{z}\left(1-\sqrt{\frac{8(z-1)}{(d+z-\theta)(1+e^2)}}\right)$$

where e^2 is defined as the ratio of the Maxwell term to one of the lattice terms in the Lagrangian. However, in [?] this formula was only justified by numerical calculation for the case $e^2 = 0$ on a neutral background. In this paper, we continue to test the validity of Eq. (5) on charged backgrounds within EMD-Axion models.

Schematically, the case $e^2 \neq 0$ can be realized by relevant currents. We numerically construct specific charged backgrounds with UV completion and compute the power law of the ratio of shear viscosity to entropy density. As a result, we show that the formula for the ratio η/s in Eq. (5), previously proposed in [?] based on scaling analysis, is indeed justified even for $e^2 \neq 0$.

In this paper, we adopt the terminology regarding '(marginally) relevant' and 'irrelevant' from [?]: a current or axion being (marginally) relevant means that the Maxwell or axion terms are of the same order in powers of the radial coordinate as the curvature term and the dilaton potential in the Lagrangian; a current or axion being irrelevant means that the Maxwell or axion terms are subleading compared to the curvature term and the dilaton potential. Roughly speaking, whether a field is (marginally) relevant or irrelevant depends on whether it strongly deforms the IR geometry.

2 EMD-Axion Model and Hyperscaling Violating Metric

In this section, we present our holographic setup and outline the logic leading to the formula for the exponent κ in Eq. (5) based on scaling analysis. The action of a general EMD-Axion model in $d+2$ spacetime dimensions reads:

$$S = \int dt\, d^d x\, dr\, \sqrt{-g}\,(R + \mathcal{L}_m)$$

with

$$\mathcal{L}_m = -(\partial\phi)^2 - \sum_{i=1}^{d}(\partial\chi_i)^2 + V(\phi) - Z(\phi)F^2$$

where χ_i $(i = 1, 2, \ldots, d)$ are axions and $J(\phi)$, $Z(\phi)$, $V(\phi)$ are coupling functions or potentials of the dilaton field ϕ.

From this action, the equations of motion can be derived as:

$$R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}T - T_{\mu\nu} = 0$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$$

$$\nabla^2\phi - J'(\phi)\sum_{i=1}^{d}(\partial\chi_i)^2 + V'(\phi) - Z'(\phi)F^2 = 0$$

$$\nabla_\nu(Z(\phi)F^{\mu\nu}) = 0$$

$$\nabla_\mu(J(\phi)\partial^\mu\chi_i) = 0, \quad i = 1, 2, \ldots, d$$

For simplicity, we only consider isotropic solutions with the following ansatz:

$$ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)\sum_{i=1}^{d}dx_i^2$$

$$\phi = \phi(r), \quad \chi_i = kx_i, \quad A = A_t(r)dt$$

where translational invariance is broken by the axions χ_i but the metric and energy-momentum tensor remain homogeneous.

As stated above, we are interested in solutions interpolating between AdS in the UV and hyperscaling violating solutions in the IR. This can be realized through the process of UV completion [?]. First, we construct a hyperscaling violation solution with running dilaton. Second, we modify the local behaviors of the potentials to graft the solution onto AdS in the UV. Generally, AdS solutions exist when the dilaton reaches the extremal point of its potential and Lorentz symmetry is maintained [?].

Let us first find hyperscaling violating solutions. When the potentials behave as:

$$V(\phi) \sim V_0 e^{\alpha\phi}, \quad J(\phi) \sim e^{\beta\phi}, \quad Z(\phi) \sim e^{\gamma\phi}$$

when $\phi \to \pm\infty$, it was found in [?, ?] that scaling solutions with hyperscaling violation exist, with metric and matter fields:

$$ds^2 = r^{\frac{2\theta}{d}} \left(-\frac{dt^2}{r^{2z}} + \frac{L^2 dr^2}{r^2} + \frac{\sum_{i=1}^{d} dx_i^2}{r^2} \right)$$

$$A = Q r^{\zeta - z} dt, \quad e^{\phi} = r^{\epsilon}, \quad \chi_i = k x_i$$

where k characterizes the scale of translational invariance breaking and ζ is called the conduction exponent [?].

The parameters $\{z, \theta, \zeta, \epsilon, k, Q\}$ in ansatz (10) are determined by the equations of motion (7). The solutions are classified into four classes [?], depending on the parameters $\{\alpha, \beta, \gamma, V_0\}$ in the potentials (9). Among them, the explicit solutions with (marginally) relevant axion are shown in Appendix A. The metric in (10) can be deformed into a black hole solution:

$$ds^2 = r^{\frac{2\theta}{d}} \left(-f(r)\frac{dt^2}{r^{2z}} + \frac{L^2 dr^2}{r^2 f(r)} + \frac{\sum_{i=1}^{d} dx_i^2}{r^2} \right)$$

where the blackness function is:

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{\delta_0}, \quad \delta_0 = d + z - \theta$$

The Hawking temperature and entropy density are given by:

$$T = \frac{z}{4\pi r_h^z}, \quad s = 4\pi r_h^{\theta - d}$$

From the Maxwell equation (7c), we have the conserved charge density:

$$\rho = \sqrt{-g} Z(\phi) F^{rt}$$

It can be shown that there exists a scaling relation:

$$x \to cx, \quad r \to cr, \quad t \to c^z t, \quad ds \to c^{\theta/d} ds, \quad T \to c^{-z} T, \quad s \to c^{d-\theta} s$$

We will return to UV completion and modify the potentials in the next section.

Now we turn to the study of shear viscosity η. From the Kubo formula (2), η can be derived by perturbing $(\delta g)_{xy}$:

$$g_{xx}\delta g_{xy} = h(r)e^{-i\omega t}$$

which is the dual field of operator $\hat{T}^{xy}$. Einstein equations (7a) give rise to the shear perturbation equation with varying mass:

$$\partial_r(\sqrt{-g}g^{rr}\partial_r h(r)) + (g^{tt}\omega^2 - m(r)^2)h(r) = 0$$

where

$$m(r)^2 = 2(g_{xx}T_{xx} - g_{tt}T_{tt})$$

The function $h(r)$ is required to be regular at the horizon and equal to 1 at the conformal boundary r_∂. The varying mass is supposed to satisfy $m(r)^2 \geq 0$ in the models considered so far. Here, we have $m(r)^2 = J(\phi)k^2 g_{xx}$.

The ratio η/s can be obtained by the weaker horizon formula [**?**, **?**]:

$$\frac{\eta}{s} = \frac{1}{4\pi}h_0(r_+)^2$$

where $h_0(r)$ is the solution at $\omega = 0$ and r_+ is the location of the horizon.

Following the analysis in [**?**], one can calculate the exponent κ of $\eta/s \sim T^\kappa$. Here we present a simpler derivation using formula (18). If the axion is (marginally) relevant, using Einstein equations (7a) and the black hole metric (11), we have:

$$m(r)^2 = \frac{M^2}{r^{\frac{2\theta}{d}}} r^{-2\delta_0(z-1)} \frac{(1+e(r)^2)}{L^2}$$

where

$$e(r)^2 = -\frac{Z(\phi)F^2}{J(\phi)(\partial\chi_x)^2} \geq 0, \quad \chi_x = kx$$

The quantity $e(r)^2$ is precisely the ratio of the Maxwell term to one of the axion terms in the Lagrangian (6). It approaches a nonzero constant in the far IR if the current is also (marginally) relevant; otherwise it goes to zero. Thus at leading order, we set $e(r)^2 = e^2$. If the axion is irrelevant, $m(r)^2$ goes to zero in the far IR, and we set $M^2 = 0$, which is valid at leading order.

Using (19) and metric (11), we rewrite the perturbation equation (16) at $\omega = 0$:

$$\partial_r(r^{1-\delta_0}f(r)\partial_r h_0(r)) - \frac{M^2}{L^2}r^{-\delta_0-1}h_0(r) = 0$$

Solving this equation, we obtain the asymptotic expansion of $h_0(r)$ near the boundary and its value on the horizon:

$$h_0(r \to r_i) = c\left[\left(\frac{r}{r_i}\right)^{\delta_0-\delta_{\hat{T}}} + \cdots + G\left(\frac{r}{r_i}\right)^{\delta_{\hat{T}}} + \cdots\right]$$

$$h_0(r_+) = cH$$

where

$$\delta_{\hat{T}} = \sqrt{\delta_0^2 + \frac{2M^2L^2}{(1+e^2)}}$$

We have abbreviated the series $\sum_{m\geq 1}(r/r_i)^{\delta_{\hat{T}}+m\delta_0}$ to ellipsis. The coefficients G and H are r_+-independent constants that depend only on δ_0 and $\delta_{\hat{T}}$.

The coefficient c should be determined by the boundary condition $h_0(r_\partial) = 1$. Here r_i is the boundary of the region where the black hole with hyperscaling violation can be described by (11). When considering the second-order variation of the action in (6) over a fixed background with the perturbed metric $h_0(r)$, the boundary part of the variation with the branch $(r/r_i)^{\delta_0-\delta_{\hat{T}}}$ is divergent, while the branch $(r/r_i)^{\delta_{\hat{T}}}$ is finite. Thus the branch $(r/r_i)^{\delta_0-\delta_{\hat{T}}}$ is non-normalizable, and the series $\sum_{m\geq 1}(r/r_i)^{\delta_{\hat{T}}+m\delta_0}$ is normalizable. Then $h_0(r_i) \approx c(r_i/r_i)^{\delta_0-\delta_{\hat{T}}}$ is a good approximation when the black hole is near-extremal.

Since UV completion is taken into account, one should be cautious that r_i is just an intermediate scale, not the conformal boundary r_∂. The region between r_i and r_∂ is AdS deformed by matter fields. The boundary condition requires $h_0(r_\partial) = 1$. As explained in [?], when evolving from r_∂ to r_i, $h_0(r)$ decreases monotonically from 1 to a value Γ when $m(r)^2 > 0$. We introduce a 'tunneling rate' Γ to characterize how $h_0(r)$ tunnels from r_∂ to r_i. The tunneling rate Γ and intermediate scale r_i should be independent of T when the temperature scale is much smaller than other scales, such as the dilaton source and k in the axion. This is because temperature is not the dominant scale driving the renormalization group (RG) flow from AdS to the hyperscaling violating solution; temperature becomes important only when we go into the far IR. Therefore:

$$\Gamma = h_0(r_i) \approx c\left(\frac{r_i}{r_i}\right)^{\delta_0-\delta_{\hat{T}}}$$

By determining c, we can find the horizon value:

$$h_0(r_+) = \Gamma H \left(\frac{r_+}{r_i}\right)^{\delta_0 - \delta_{\hat{T}}}$$

Using formula (18), we finally obtain:

$$\frac{\eta}{s} \sim \Gamma^2 H^2 \left(\frac{r_+}{r_i}\right)^{2(\delta_0 - \delta_{\hat{T}})} \sim T^{\frac{d-\theta+z}{z}\left(1-\sqrt{\frac{8(z-1)}{(d-\theta+z)(1+e^2)}}\right)}$$

when $T \to 0$, where (13) and (23) have been used. One can also employ UV-IR matching [**?**, **?**, **?**] to reproduce the same result, as performed in [**?**]. As can be seen, when the axion is absent or irrelevant, we have $m(r)^2 = 0$ at leading order, and $\eta/s \sim T^0$ [**?**, **?**]. Next, we will numerically test this formula for both $e^2 = 0$ and $e^2 \neq 0$.

3 UV Completion and Numerical Results

In this section, we specifically construct backgrounds that interpolate between the hyperscaling violating solution (10) in the IR and the AdS solution in the UV, at finite temperature and charge density. On one hand, as explained in Appendix A or in [**?**], solution (10) can be constructed by choosing exponential potentials (9) with running dilaton. On the other hand, AdS can be constructed by finding extremal points of constant dilaton with Lorentz symmetry. Here we adopt the dilaton ϕ to interpolate between the UV and IR solutions, which requires special settings for the potential $V(\phi)$.

3.1 UV Completion

In this paper, we focus only on $\theta < d$ since in this region the entanglement entropy obeys an area-to-volume law, which is considered normal behavior for quantum field theories [**?**, **?**]. In this situation, the IR region is at $r \to +\infty$. Following the discussion in [**?**, **?**], the constraints on (z, θ) reduce to:

$$(\theta \leq 0 \wedge z > 1) \vee \left(0 < \theta < d \wedge z > 1 + \frac{\theta}{d}\right)$$

This leads to $\delta_0 > 0$. We have excluded the two cases $\theta = d$ and $z = \theta/d + 1$, which cannot be reached by running dilaton, as shown in Appendix A or [**?**]. We choose the branch $\phi \geq 0$. From the requirement of the potentials (9), one can see that our solution can flow to the hyperscaling violating solution in the IR if $\phi \to +\infty$ $(r \to +\infty)$. This requires $\epsilon > 0$ in (10). We perform UV completion by modifying the potential $V(\phi)$ while fixing the other two coupling potentials as $J(\phi) = e^{\beta\phi}$ and $Z(\phi) = e^{\gamma\phi}$.

From the analysis in Appendix A, we find a universal behavior $V(\phi) \sim r^{-2\theta/d}$ in the coordinates of ansatz (10). Therefore, when approaching the UV $(r \to 0)$, the qualitative behavior of $V(\phi)$ depends on the sign of θ. In the UV region, AdS is allowable if the axion and gauge field are turned off and $\phi = \phi_*$ is an extremal point of $V(\phi)$, where $V(\phi_*)$ acts as the cosmological constant. Without loss of generality, we choose $\phi_* = 0$. A realistic strategy is to modify $V(\phi)$ as:

$$V(\phi) = \begin{cases} \alpha^2 \sinh^2\left(\frac{\alpha\phi}{\sqrt{2d(d+1)}}\right) + V_0 e^{\alpha\phi}, & \text{for } \theta < 0 \\ \frac{d(d+1)}{2}\cosh(\alpha\phi) + V_0 e^{\alpha\phi}, & \text{for } \theta = 0 \\ \frac{d(d+1)}{\cosh^2(\alpha\phi)} + V_0 e^{\alpha\phi}, & \text{for } d > \theta > 0 \end{cases}$$

with

$$V_0 = \begin{cases} \frac{2d}{\alpha^2}\left(1 + \frac{2d+2}{\alpha^2}\right), & \text{for } \theta < 0 \\ \frac{d}{2\alpha^2}, & \text{for } \theta = 0 \\ (d+1)d, & \text{for } d > \theta > 0 \end{cases}$$

In Appendix A, we have $\alpha\epsilon = -2\theta/d$, so α has the opposite sign of θ. As can be seen, each $V(\phi)$ approaches $V_0 e^{\alpha\phi}$ when $\phi \to +\infty$ and approaches $d(d+1)$ when $\phi \to 0$. Without loss of generality, we have chosen the AdS radius to be 1. However, the intermediate behavior is not crucial.

The AdS_{d+2} vacuum is always allowable. We choose the first-type quantization, and the scaling dimensions of the dual source of dilaton $\phi_{(0)}$ and operator $\mathcal{O}_\phi$ are determined by the small-ϕ expansion of $V(\phi)$. When $\theta = 0$, as the dilaton is massless, we have $\Delta_{\phi_{(0)}} = 0$ and $\Delta_{\mathcal{O}_\phi} = d + 1$, which is a marginal deformation. We expect it to be marginally relevant to drive the solution away from AdS, as $\phi = 0$ is not a stable point when axion and gauge field are turned on. When $\theta \neq 0$, as $V(\phi) = d(d+1) + \frac{d}{2}\phi^2 + \cdots$, we have $\Delta_{\phi_{(0)}} = 1$ and $\Delta_{\mathcal{O}_\phi} = d$, which is a relevant deformation.

3.2 Numerical Calculation and Results

We use the following ansatz for numerical calculation:

$$ds^2 = \frac{1}{u^2}\left[-(1-u)U(u)e^{-S(u)}dt^2 + \frac{du^2}{4(1-u)U(u)} + \sum_{i=1}^{d} dx_i^2\right]$$

$$\phi = \phi(u), \quad \chi_i = kx_i, \quad A = (1-u)A(u)dt$$

The conformal boundary is located at $u = 0$ while the horizon is at $u = 1$. The temperature and entropy density are:

$$T = \frac{1}{4\pi}U(1)e^{-S(1)/2}, \quad s = 4\pi$$

The AdS$_{d+2}$ vacuum corresponds to $U = 1$, $S = \phi = A = 0$. Boundary conditions at the horizon are regularity conditions. Boundary conditions at the conformal boundary should satisfy the scaling dimensions, which depend on the potential $V(\phi)$ as well as the value of θ.

Explicitly, the asymptotic expansions near the conformal boundary are:

$$U(u) = 1 + \cdots + \epsilon u^{d+1} + \cdots$$

$$e^{-S(u)} = 1 + \cdots$$

$$A(u) = \mu + \cdots + \rho u^{d-1} + \cdots$$

$$\phi(u) = \begin{cases} \lambda + \cdots + \nu u^{d+1} + \cdots, & \theta = 0 \\ \lambda u + \cdots + \nu u^{d} + \cdots, & \theta \neq 0 \end{cases}$$

where μ is the chemical potential and $e^{-S(0)}$ has been set to 1 by rescaling t. The different boundary conditions for $\phi(u)$ come from the different choices of $V(\phi)$ in (29). We can work in either the grand canonical ensemble or the canonical ensemble.

3.2.1 Grand Canonical Ensemble In the grand canonical ensemble, we control the value of the chemical potential μ.

When $\theta = 0$, the boundary conditions at the conformal boundary are $U(0) = 1$, $S(0) = 0$, $A(0) = \mu$, $\phi(0) = \lambda$. We work in units of k. The dimensionless quantities parameterizing the family of black hole solutions are $\{T/k, \mu/k, \lambda\}$.

When $\theta \neq 0$, the boundary conditions are $U(0) = 1$, $S(0) = 0$, $A(0) = \mu$, $\phi'(0) = \lambda$. The dimensionless quantities are $\{T/k, \mu/k, \lambda\}$.

We numerically construct the interpolating solutions for $\phi \geq 0$. When lowering T/k, we should fix the values of $\{\mu/k, \lambda\}$ (for $\theta = 0$) or $\{\mu/k, \lambda\}$ (for $\theta \neq 0$) within an appropriate region to reach the hyperscaling violating solution in the IR at low T/k.

The dimensionless entropy density and charge density are s/k^d and ρ/k^d. We calculate η/s using (18) and find $\eta/s \leq 1/4\pi$ at all times, due to the breaking of translational invariance.

At high T/k, the scaling relation is controlled by AdS in the UV, giving power laws $s \sim T^d$ and $\eta/s \sim T^0$. On the other hand, at low T/k, hyperscaling violation emerges in the IR. The power laws $s \sim T^{(d-\theta)/z}$ and $\eta/s \sim T^{\kappa}$ are observed in numerical results.

It is worth noting that ρ/k^d converges to a nonzero constant at low T/k. When approaching the IR, $e^2(u)$ converges to a nonzero constant for Class I but goes to zero as some power of the radial coordinate u for Class II.

Similarly, at low T/k, the horizon value $e_h^2 = e^2(1)$ converges to a nonzero constant for Class I but goes to zero as a power law for Class II, with the exponent shown in Appendix A. For the same $\{d, z, \theta\}$, it is observed that the appearance of a nonzero e^2 always reduces the exponent κ of $\eta/s \sim T^\kappa$, consistent with the property that κ decreases monotonically with e^2 in (26) when $\kappa > 0$.

We conduct numerical calculations for $d = 2$ and $\theta = 4/3, 0, -4$ as representatives of three cases: $\theta < 0$, $\theta = 0$, and $0 < \theta < d$. Different values of γ are chosen to represent Class I or Class II. The specific results are shown in Figure 1 [Figure 1: see original paper], Figure 2 [Figure 2: see original paper], and Figure 3 [Figure 3: see original paper]. All numerical results match the analytical predictions.

3.2.2 Canonical Ensemble In the canonical ensemble, we control the value of charge density ρ. The t-component of Maxwell' s equations (7c) can be replaced by (14).

When $\theta = 0$, the three-parameter family of solutions is characterized by $\{T/k^d, \rho/k^d, \lambda\}$. When $\theta \neq 0$, it is characterized by $\{T/k, \rho/k^d, \lambda\}$. When lowering T/k, we fix the other two dimensionless parameters within an appropriate region. Then μ/k converges to a nonzero constant at low T/k (instead of ρ/k^d as in the grand canonical ensemble). The behaviors of η/s and e^2 are similar to those in the grand canonical ensemble. Using the method in Appendix A, we can predict the value to which e^2 converges in the IR at low temperature for Class I.

Similarly, we conduct numerical calculations for $d = 3$ and $\theta = -6, 0, 2$ to represent the three regions of $\theta < 0$, $\theta = 0$, and $0 < \theta < d$. The specific results are shown in Figure 4 [Figure 4: see original paper], Figure 5 [Figure 5: see original paper], and Figure 6 [Figure 6: see original paper]. All numerical results match the analytical predictions.

4 Conclusion and Outlook

In this paper, we have numerically constructed charged solutions with emerging hyperscaling violation in the EMD-Axion model and investigated the temperature behavior of the ratio of shear viscosity to entropy density. We have found that the relevant axion, which breaks translational invariance, leads to the power law $\eta/s \sim T^\kappa$. In particular, the relevant current reduces the exponent κ indeed. This reduction is characterized by the quantity e^2, which can be derived from the dimensionless conserved charge density ρ/k^d. Meanwhile, irrelevant current does not affect the exponent κ since $e^2 \to 0$ in the far IR at low temperature.

Our analytical results for the exponent κ coincide with our numerical calculations, indicating that our proposed formula for κ in (5) is robust at least for generic backgrounds within the EMD-Axion model.

Especially, our results for the Lifshitz case verify that $\kappa > 2$ can occur for $d > 2$, indicating that hyperscaling violation is not an essential ingredient for obtaining $\kappa > 2$. Moreover, as conjectured in [?], the upper bounds for κ coincide with the behaviors of entanglement entropy.

Analytically, it is possible that irrelevant current or axion can affect the temperature behavior of η/s at subleading orders. One should consider their backreaction to the background and then solve the shear perturbation equation (16). This is related to the issue of whether temperature T remains the unique scale in entropy production, as it is when the axion is relevant. This warrants future investigation.

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Appendix A: Classification of IR Solutions

We focus on relevant axion solutions; otherwise η/s simply converges to a nonzero constant at low temperature at leading order. We require the solutions to have positive specific heat, and the temperature deformation must be the only allowed relevant deformation. Additionally, we impose the requirement $\theta < d$. The scaling solutions have been obtained and classified in [?, ?].

A.1 Class I: Marginally Relevant Current, Marginally Relevant Axion

If both the Maxwell and axion terms are of the same order in powers of the radial coordinate as the curvature term and the dilaton potential $V(\phi)$ in the Lagrangian (6), the scaling solutions form a one-parameter family:

$$\beta\epsilon = -2, \quad \alpha\epsilon = -\frac{2\theta}{d}$$

$$\gamma = \alpha(d-1) - \beta d, \quad \epsilon^2 = \frac{2(d-\theta)(d(z-1)-\theta)}{\delta_0}$$

$$Q^2 = \frac{2(k^2(dz-\theta) + 2V_0(1-z))}{\delta_0((d-1)k^2 - 2V_0)}, \quad \zeta = \theta - d$$

$$L^2 = \frac{2V_0 - (d-1)k^2}{\delta_0(\delta_0 - 1)}$$

which can be parameterized by k in the coordinates of (10). The charge-related quantities are:

$$\rho^2 = \frac{k^2(\theta - dz) + 2V_0(z-1)}{\delta_0 - 1}$$

$$e^2 = \frac{k^2(\theta - dz) + 2V_0(z-1)}{k^2(\delta_0 - 1)}$$

The mode analysis in [?] indicates that there are three pairs of conjugate modes summing to δ_0. Two pairs are degenerate with $\beta_{1,-} = \beta_{2,-} = 0$ and $\beta_{1,+} = \beta_{2,+} = \delta_0$. The mode $\beta_{1,-}$ rescales time. The mode $\beta_{1,+}$ is the temperature deformation responsible for creating a small black hole (11). The mode $\beta_{2,-}$ changes k and shifts the solution along the one-parameter family. The mode $\beta_{2,+}$ changes the chemical potential and belongs to gauge symmetry transformations. The expression for the last pair $\beta_{3,\pm}$ is too tedious to show here. We require that $\beta_{1,+}$ is relevant and $\beta_{3,-}$ is irrelevant, with the IR located at $r \to \infty$. The final allowed parameter space is found to be $\rho^2 > 0$ and (27).

Since the quantity ρ is conserved and invariant under coordinate transformations within (8), we can use it to connect UV with IR and determine the solution in the one-parameter family using (33). Finally, e^2 can be obtained from ρ/k^d using (34), which is convenient in the canonical ensemble.

At zero temperature, one can integrate the three modes $\beta_{2,-}$, $\beta_{2,+}$, and $\beta_{3,-}$ to the UV and adjust them to satisfy the boundary conditions specified by $\{\lambda, \rho, \mu\}$ at the conformal boundary. A finite-temperature solution can be driven by $\beta_{1,+}$.

A.2 Class II: Irrelevant Current, Marginally Relevant Axion

If only the Maxwell term becomes subleading, a single scaling solution at leading order is obtained:

$$\beta\epsilon = -2, \quad \alpha\epsilon = -\frac{2\theta}{d}$$

$$\epsilon^2 = \frac{2(d-\theta)(d(z-1)-\theta)}{\delta_0(dz-\theta)}, \quad Q^2 = \frac{2V_0(z-1)}{dz-\theta}$$

There are three pairs of modes, where two pairs sum to δ_0. The first pair is $\beta_{1,-} = 0$ and $\beta_{1,+} = \delta_0$, corresponding to time rescaling and temperature deformation. The second pair consists of relevant $\beta_{2,+}$ and irrelevant $\beta_{2,-}$. The third pair comprises gauge field modes with $A(r) = A_1 + A_2 r^{\zeta - z}$, where $\zeta = d - \theta + \frac{\epsilon\gamma}{2}$.

The mode A_2 is irrelevant when $\delta_0(\zeta + d - \theta) < 0$. We find:

$$e^2(r) \sim r^{\zeta - d + \theta}, \quad \rho \sim T^0, \quad h \sim T^{\zeta - d + \theta}, \quad \text{when } T \to 0$$

Similar to Class I, we can integrate $\beta_{2,-}$ and gauge field modes to the UV.

Note: Figure translations are in progress. See original paper for figures.