Perturbation Analysis Method for the Impact of Satellite Orbit Errors on Positioning Accuracy: Postprint

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Abstract

The positioning accuracy of satellite navigation systems is comprehensively influenced by multiple factors, including pseudorange measurement errors, atmospheric delay errors, satellite atomic clock biases, and satellite orbit errors. Traditionally, evaluation of positioning errors typically employs methods based on Dilution of Precision (DOP) and User Equivalent Range Error (UERE). However, the derivation of their accuracy characterization formulas requires several assumptions regarding the coefficient matrix H of the measurement equation system and the error distribution of the User Equivalent Range Error; thus, it is essentially an approximate evaluation formula. Moreover, among various error sources, satellite orbit errors are three-dimensional errors that require coordinate transformation and empirical parameter models to be converted into User Equivalent Range Error. To address this issue, we propose employing matrix perturbation theory to investigate the effect of satellite orbit errors on the solution of the positioning equation system, using the spectral norm condition number to characterize the structure of the equation system. Simulation results indicate that the proposed method can directly reflect the impact of satellite orbit errors on positioning accuracy without requiring orbit coordinate conversion or User Equivalent Range Error transformation, thereby enabling a more direct and accurate assessment of the influence of satellite orbit errors on positioning solution accuracy.

Full Text

Preamble

Perturbation Analysis Method for the Influence of Satellite Orbit Error on Positioning Accuracy

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Abstract

GNSS positioning accuracy is comprehensively affected by multiple factors including pseudo-range measurement errors, atmospheric delay errors, satellite atomic clock biases, and satellite orbit errors. Traditionally, the Dilution of Precision (DOP) and User Equivalent Range Error (UERE) parameters are used to estimate positioning error. However, the derivation of the precision characterization formula requires several assumptions about the measurement equation coefficient matrix H and the distribution of UERE, making it essentially an approximate evaluation formula. Furthermore, among various error sources, satellite orbit error constitutes a three-dimensional error that requires coordinate transformation and empirical parameter models to convert into UERE. To address this limitation, this paper proposes using matrix perturbation theory to study the influence of satellite orbit error on the positioning equation solution, employing the spectral norm condition number to characterize the equation system' s morphology. Simulation results demonstrate that this method can directly reflect the impact of satellite orbit error on positioning accuracy without requiring orbit coordinate conversion or UERE translation, enabling more direct and accurate assessment of how satellite orbit error affects positioning solution precision.

Keywords: Satellite Orbit Error; Navigation and Positioning; User Equivalent Range Error; Norm; Condition Number

Introduction

Global Navigation Satellite System (GNSS) positioning error is typically characterized by the combination of Dilution of Precision (DOP) and User Equivalent Range Error (UERE), expressed as:

$$\sigma_U = \text{DOP} \cdot \sigma_{UERE} \quad (1)$$

For a given satellite, the User Equivalent Range Error is treated as the statistical sum of contributions from various error sources associated with that satellite, including pseudo-range measurement errors, atmospheric delay errors, satellite atomic clock biases, and satellite orbit errors.

Throughout the development of GNSS systems, this error evaluation methodology has played a crucial role. It serves not only as an important basis for constellation system design but also as a key indicator for users to perform constellation optimization and positioning accuracy prediction and analysis. Depending on the type of DOP parameter, users can rapidly evaluate the solution accuracy of unknown parameters in different coordinate directions as well as receiver clock bias.

However, for the sake of convenience, the DOP/UERE-based precision characterization method has several theoretical limitations. First, the precision characterization formula is derived based on two assumptions: (1) the observation equation coefficient matrix H has no random components, allowing it to be moved outside the expectation operator during derivation; and (2) all UEREs share the same variance and are uncorrelated zero-mean errors, enabling simplification of the expectation operator. However, the coefficient matrix H only becomes deterministic when all observed satellites are orthogonally distributed, while the composition of UERE is highly complex, encompassing not only random measurement errors following Gaussian white noise but also various atmospheric delays, hardware-induced biases, and multipath effects at the receiver. Consequently, both assumptions are difficult to satisfy in practical positioning scenarios.

Second, among various positioning error sources, pseudo-range measurement errors, atmospheric delay errors, and satellite atomic clock biases are one-dimensional vectors—errors along the line-of-sight direction from satellite to user—that can be easily converted into UERE. However, satellite orbit error (generally considered as satellite broadcast ephemeris error) is a three-dimensional error quantity that cannot be directly converted into UERE. To assess its impact on positioning accuracy, the satellite broadcast ephemeris error must typically be transformed from the Earth-Centered Earth-Fixed (ECEF) coordinate system to the orbital coordinate system (RTN frame) at that epoch, obtaining deviations in the radial (R), along-track (T), and cross-track (N) directions. Based on this transformation, the radial component is considered to have the greatest impact on user positioning accuracy, and empirical models with relevant parameters are then used to calculate the satellite User Range Error (URE) value for incorporation into the comprehensive UERE consideration. For hybrid constellations like China' s BeiDou Navigation Satellite System (BDS) with MEO, GEO, and IGSO satellites, different empirical parameters must be applied to different orbit types, increasing the complexity and cumbersome nature of the precision analysis and evaluation process.

To address these challenges, this paper proposes employing matrix perturbation theory to investigate and analyze the influence of satellite orbit error on the positioning observation equation solution, using the spectral norm-based condition number to characterize the measurement equation system' s morphology. In fact, the mathematical essence of DOP is precisely the Frobenius norm (F-norm) of a matrix. Simulation results demonstrate that matrix perturbation theory can directly reflect the impact of satellite orbit error on positioning accuracy without requiring equivalent pseudo-range error conversion, enabling more direct and accurate investigation of positioning equation system performance, solution errors, and the influence of satellite orbit error on positioning solution precision.

2 Observation Equations and Error Equations

The linearized matrix form of the GNSS single-point positioning solution can be expressed as:

$$\mathbf{H}\Delta\mathbf{X} = \Delta\rho \quad (2)$$

where $\mathbf{H}$ is the direction cosine matrix:

$$\mathbf{H} = \begin{bmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 & 1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cos\alpha_n & \cos\beta_n & \cos\gamma_n & 1 \end{bmatrix} \quad (3)$$

with $\cos\alpha_i$, $\cos\beta_i$, $\cos\gamma_i$ representing the direction cosines of the unit vector from the user estimated position to the i-th satellite.

$\Delta\mathbf{X}$ is the correction vector between the approximate solution and true solution:

$$\Delta\mathbf{X} = [\Delta x, \Delta y, \Delta z, -\Delta t_c]^T \quad (4)$$

and $\Delta\rho$ is the difference between the geometric range calculated from user estimated position and clock bias and the observed pseudo-range:

$$\Delta\rho = [\Delta\rho_1, \Delta\rho_2, \dots, \Delta\rho_n]^T \quad (5)$$

When $n = 4$, equation (2) is a determined system with solution:

$$\Delta\mathbf{X} = \mathbf{H}^{-1}\Delta\rho \quad (6)$$

When $n > 4$, equation (2) is an over-determined system. The least-squares solution is:

$$\Delta\mathbf{X} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\Delta\rho \quad (7)$$

According to the derivation of the covariance matrix for least-squares solutions in references [1-3], the covariance of user coordinate position error is:

$$\mathrm{cov}(\Delta\mathbf{X}) = E[(\Delta\mathbf{X}-E[\Delta\mathbf{X}])(\Delta\mathbf{X}-E[\Delta\mathbf{X}])^T] = \begin{cases} (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T E[\Delta\rho\Delta\rho^T]\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}, & n > 4 \\ \mathbf{H}^{-1}E[\Delta\rho\Delta\rho^T](\mathbf{H}^{-1})^T, & n = 4 \end{cases} \quad (8)$$

where E is the expectation operator, σ_u is the user position error standard deviation, and $\Delta\rho$ is the pseudo-range observation error vector.

In the above derivation, the system matrix $\mathbf{H}$ was assumed to have no random components and could thus be moved outside the expectation operator. Further assuming all user equivalent pseudo-range errors share the same variance σ_ρ^2 and are uncorrelated zero-mean errors, the expectation operator in equation (8) simplifies to:

$$E[\Delta\rho\Delta\rho^T] = \sigma_\rho^2\mathbf{I} \quad (9)$$

where $\mathbf{I}$ is the 4×4 identity matrix. Consequently, the two equations in (8) simplify to:

$$\mathrm{cov}(\Delta\mathbf{X}) = \sigma_\rho^2(\mathbf{H}^T\mathbf{H})^{-1} \quad (10)$$

The DOP represents the error amplification factor when converting user equivalent pseudo-range error to user position error, which is inversely proportional to the solid volume delineated by the user-to-satellite unit vectors. Depending on the user' s coordinate reference frame, DOP can characterize position errors in different directions. When the user is in the user equivalent pseudo-range error coordinate system, its actual meaning is:

$$\mathrm{cov}(\Delta\mathbf{X}) = (\mathbf{H}^T\mathbf{H})^{-1} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} & \sigma_{xt} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} & \sigma_{yt} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 & \sigma_{zt} \\ \sigma_{tx} & \sigma_{ty} & \sigma_{tz} & \sigma_t^2 \end{bmatrix} \quad (11)$$

The square roots of its diagonal elements correspond to the error amplification factors (DOP values) for the three coordinate position components and receiver clock bias, respectively.

Since the derivation of DOP assumes the coefficient matrix $\mathbf{H}$ has no random components, the covariance matrix of $\mathbf{H}$ in equation (11) is considered zero. This reveals that DOP' s definition is consistent with the Frobenius norm (F-norm) of a matrix—the square root of the sum of squares of all matrix elements. Thus, the mathematical essence of DOP is the matrix F-norm, demonstrating that matrix norm perturbation theory can be further employed to conduct in-depth research on positioning error solutions of measurement equation systems.

3 Orbit Error Perturbation Analysis Method

The above error equation derivation shows that the DOP-based analysis method makes conditional assumptions about both the statistical characteristics of measurement errors and the random components of the coefficient matrix $\mathbf{H}$ (assuming measurement errors follow Gaussian distributions and are independent, and that system matrix $\mathbf{H}$ has no random components). In practice, however, the satellite navigation positioning process rarely satisfies these ideal conditions. In contrast, norm-based perturbation analysis theory can study the actual morphology of measurement equation systems and provide more direct and realistic assessment of satellite orbit error impact.

When investigating the influence of satellite orbit error on the positioning solution, we consider that perturbation errors $\delta\mathbf{H}$ in the left-hand coefficient matrix $\mathbf{H}$ cause perturbations $\delta\Delta\mathbf{X}$ in the unknown coordinate and clock bias vector $\Delta\mathbf{X}$. This gives:

$$(\mathbf{H}+\delta\mathbf{H})(\Delta\mathbf{X}+\delta\Delta\mathbf{X})=\Delta\rho \quad (12)$$

It can be proven that since matrix $\mathbf{H}$ is non-singular and the perturbation errors caused by satellite orbit errors are sufficiently small, the perturbed matrix $\mathbf{H}+\delta\mathbf{H}$ can maintain its non-singularity. Subtracting equation (2) from equation (12) yields:

$$\delta\Delta\mathbf{X}=-(\mathbf{H}+\delta\mathbf{H})^{-1}\delta\mathbf{H}\Delta\mathbf{X} \quad (13)$$

Applying norm expressions to both sides of equation (12) gives:

$$\|\delta\Delta\mathbf{X}\|\leq\|(\mathbf{H}+\delta\mathbf{H})^{-1}\|\|\delta\mathbf{H}\|\|\Delta\mathbf{X}\| \quad (14)$$

Further derivation leads to:

$$\frac{\|\delta\Delta\mathbf{X}\|}{\|\Delta\mathbf{X}\|}\leq\|\mathbf{H}^{-1}\|\|\mathbf{H}\|\frac{\|\delta\mathbf{H}\|}{\|\mathbf{H}\|} \quad (15)$$

where $\|\mathbf{H}^{-1}\|\|\mathbf{H}\|$ is the error amplification factor—the ratio of relative error in the solution to relative error in the measurement data. This quantity is known as the matrix condition number, denoted as:

$$\mathrm{cond}(\mathbf{H})=\|\mathbf{H}^{-1}\|\|\mathbf{H}\| \quad (16)$$

Thus, equation (15) can be written as:

$$\frac{\|\delta\Delta\mathbf{X}\|}{\|\Delta\mathbf{X}\|}\leq\mathrm{cond}(\mathbf{H})\frac{\|\delta\mathbf{H}\|}{\|\mathbf{H}\|} \quad (17)$$

Since $\|\delta\mathbf{H}\|/\|\mathbf{H}\|$ is generally sufficiently small, equation (17) can typically be approximated as:

$$\frac{\|\delta\Delta\mathbf{X}\|}{\|\Delta\mathbf{X}\|} \leq \operatorname{cond}(\mathbf{H})\frac{\|\delta\mathbf{H}\|}{\|\mathbf{H}\|} \quad (18)$$

This demonstrates that when $\mathbf{H}$ has perturbation $\delta\mathbf{H}$, the resulting relative error in the solution does not exceed $\operatorname{cond}(\mathbf{H})$ times the relative error in $\mathbf{H}$.

The above analysis shows that when perturbation errors exist in the coefficient matrix $\mathbf{H}$ of an equation system, the solution error is determined by the condition number, which acts as an error transmission amplification factor similar to DOP. The condition number depends on the chosen matrix norm, and mathematically, the spectral norm (2-norm) is typically used for characterization:

$$\operatorname{cond}(\mathbf{H}) = \|\mathbf{H}^{-1}\|_2\|\mathbf{H}\|_2 = \sqrt{\frac{\lambda_{\max}(\mathbf{H}^T\mathbf{H})}{\lambda_{\min}(\mathbf{H}^T\mathbf{H})}} \quad (19)$$

This is also known as the matrix spectral condition number. The condition number simultaneously characterizes the equation system' s morphology and reflects the solution' s sensitivity to errors. When $\operatorname{cond}(\mathbf{H})$ is relatively large, the measurement equation system is called ill-conditioned; when $\operatorname{cond}(\mathbf{H})$ is relatively small, it is called well-conditioned.

4 Simulation Experiments and Analysis

To analyze and verify the feasibility of the norm-based perturbation analysis method, a combined approach using real measurements and simulations was employed. First, a GNSS receiver was used to collect GPS satellite orbit coordinate data over a period of epochs at a fixed location with known coordinates. To facilitate error isolation and control, corresponding GPS pseudo-range data were simulated with 1 = 3 m pseudo-range errors. Second, to investigate the impact of satellite orbit error on positioning accuracy, 1 = 3 m orbit coordinate errors were generated and added in each of the X, Y, and Z directions (resulting in a three-dimensional orbit error of 1 = 5.196 m). By calculating positioning results before and after adding orbit errors, the actual solution errors caused by satellite orbit errors were obtained and compared with errors estimated by the norm-based perturbation analysis method, as shown in Figure 1 [Figure 1: see original paper].

Figure 1 Comparison between perturbation analysis method error estimates and actual solution errors

The results demonstrate that based on matrix norm perturbation theory, the influence of satellite orbit errors on GNSS positioning solutions can be directly evaluated and analyzed without relying on error distribution assumptions or undergoing orbit coordinate decomposition and transformation. Moreover, the error upper bounds calculated by the norm perturbation analysis method consistently lie above the actual solution errors, with good agreement between them (differences remain within a few meters), indicating that the norm perturbation analysis method can accurately and reliably characterize and evaluate GNSS positioning solution errors.

The DOP/UERE-based precision characterization method has played an important role in constellation optimization, positioning accuracy prediction, and analysis. However, since the DOP/UERE precision characterization formula is derived based on several assumptions, it can only serve as an approximate evaluation method in the strict sense and remains theoretically inadequate. Additionally, converting all positioning error sources into the UERE concept for evaluation introduces difficulties and increases the complexity of assessing the impact of satellite orbit errors on user positioning accuracy.

This paper proposes a new theoretical and practical approach based on norm mathematics and matrix perturbation analysis to study the influence of satellite orbit errors, enriching and complementing the traditional DOP/UERE-based precision characterization method. When prior statistical information about satellite orbit errors is known, this method can be used to predict and evaluate the degree of influence of satellite orbit errors. Alternatively, after obtaining true satellite orbit errors from post-processed precise ephemerides, it enables more accurate and convenient analysis of satellite constellation system performance. This method avoids conditional assumptions about measurement error statistics and the random components of the observation equation coefficient matrix, requires no analytical decomposition or coordinate transformation of satellite orbit errors, and does not rely on statistical significance, allowing detailed description of error perturbations in single positioning solutions. Therefore, it provides a more comprehensive and direct evaluation and analysis of how satellite orbit errors affect user positioning accuracy. Simulation experiments and analysis demonstrate that the matrix norm-based perturbation analysis method features concise calculation processes and provides accurate and reliable evaluation results, possessing both theoretical significance and practical application value.

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Note: Figure translations are in progress. See original paper for figures.