

Postprint: Fast Dynamic Adaptive Wavelet Collocation Method Based on Precise Time Integration

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Abstract

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Full Text

A Fast Dynamically Adaptive Wavelet Collocation Method Based on Precise Time-Integration

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Abstract

The dynamically adaptive wavelet collocation method (AWCM) is a novel numerical technique developed in recent years for efficiently simulating physical phenomena with intermittency. To enhance its ability to identify and track solution

singularities while improving numerical stability and computational efficiency, this paper combines the precise time-integration algorithm with AWCM to establish a fast dynamically adaptive multiscale wavelet collocation method. To implement this algorithm, we propose a new approach for constructing dynamic adaptive grid collocation sets and develop time-marching formulas with wavelet (or scaling function) coefficients as variables. By solving the one-dimensional Burgers equation, we demonstrate that the proposed method exhibits superior numerical properties.

Keywords: Wavelet; Dynamically adaptive grids; Precise time-integration; Intermittency

Introduction

Wavelet methods constitute a remarkably innovative research area in computational fluid dynamics. The earliest work can be traced back to 1990, when Glowinski et al. [?] successfully applied wavelet methods to solve elliptic and hyperbolic partial differential equations. Vasilyev has made substantial contributions to this field. In 1995, he proposed the multiscale wavelet collocation method based on collocation methods for solving PDEs and wavelet multiresolution analysis [?]. In 1996, building upon this foundation, he developed the dynamically adaptive multiscale wavelet collocation method by leveraging the wavelet's capability for detecting signal singularities [?]. By 2000, he had constructed a second-generation wavelet collocation method for solving partial differential equations [?].

In turbulence flow simulation using wavelet methods, the earliest approach was wavelet-based direct numerical simulation (WDNS), which fully considered the intermittency of turbulent flow to reduce computational complexity. Subsequently, in 1999, Farge et al. [?] proposed the coherent vortex simulation (CVS) method, which decomposes turbulent flow into coherent and random background components using wavelets—the former being directly computed while the latter is modeled with simplified equations. In 2004, Goldstein et al. [?] introduced the stochastic coherent adaptive large eddy simulation (SCALES) method, which inherits CVS's ability to track high-energy coherent structures in turbulent flow while employing large eddy simulation (LES) to model low-energy random fluctuations.

In 1994, Academician Zhong Wanxie [?] proposed the precise time-integration method for structural dynamics. This algorithm employs approximate discretization only in space, making it a semi-analytical numerical method for linear problems. By precisely computing the constant exponential matrix used for time-level advancement, it achieves high temporal accuracy. This paper synthesizes considerations of numerical computational efficiency and stability to propose a fast dynamically adaptive multiscale wavelet collocation method based on precise time-integration.

1 Calculation Method

This work employs the dynamically adaptive multiscale wavelet collocation method based on interpolation wavelet transforms. Since wavelet transforms can only be performed in the finite-energy function space $L^2(f)$, this discussion is restricted to functions in $L^2(f)$. Using the one-dimensional Burgers equation as an example, its dimensionless governing equation is:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{Re} \frac{\partial^2 u}{\partial x^2}$$

1.1 Interpolation Wavelet Transform for Discretizing Partial Differential Equations The wavelet transform of $u(x)$ is performed as follows:

$$u(x) = \sum_{k=0}^{N_0-1} c_{0k} \phi_{0k}(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{N_j-1} d_{jk} \psi_{jk}(x)$$

where the subscript J denotes the finest scale. The derivatives of the original function $u(x)$ with respect to x can be calculated using the derivatives of the scaling functions ϕ_k and wavelet functions ψ_k :

$$\left. \frac{\partial u}{\partial x} \right|_{x_n} = \sum_{k=0}^{N_0-1} c_{0k} \phi'_{0k}(x_n) + \sum_{j=0}^{J-1} \sum_{k=0}^{N_j-1} d_{jk} \psi'_{jk}(x_n)$$

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_n} = \sum_{k=0}^{N_0-1} c_{0k} \phi''_{0k}(x_n) + \sum_{j=0}^{J-1} \sum_{k=0}^{N_j-1} d_{jk} \psi''_{jk}(x_n)$$

Here, $u(x_n, t)$ represents the approximate value of u at the n -th collocation point on the finest-scale collocation set; c_{0k} denotes the scaling function coefficient at the k -th collocation point on the coarsest-scale set; $\phi'_{0k}(x_n)$ and $\phi''_{0k}(x_n)$ are the first and second derivatives of the scaling function at the n -th collocation point; d_{jk} represents the wavelet coefficient at the k -th collocation point on scale j ; and $\psi'_{jk}(x_n)$ and $\psi''_{jk}(x_n)$ are the first and second derivatives of the wavelet function at the n -th collocation point on the finest scale J .

1.2 Multiscale Interpolation Wavelet Collocation Method Based on Precise Time-Integration Substituting the above expressions into the governing equation yields the semi-discrete form of the one-dimensional Burgers equation:

$$\frac{dU}{dt} = -U \odot (S_1 C) + \frac{1}{Re}(S_2 C)$$

where vector C consists of wavelet (or scaling function) coefficients, and vectors S_1 and S_2 contain the first and second derivatives of wavelets (or scaling functions), respectively. This semi-discrete equation can be written in matrix form as:

$$\frac{dU}{dt} = \text{diag}(U)S_1 C + \frac{1}{Re}S_2 C$$

The relationship between vector U and vector C in the semi-discrete equation is given by:

$$U = S_0 C$$

where vector S_0 consists of wavelet (or scaling function) values. Matrix S_0 is full-rank and invertible, yielding:

$$C = S_0^{-1}U$$

Substituting this into the semi-discrete equation gives:

$$\frac{dU}{dt} = \text{diag}(U)S_1 S_0^{-1}U + \frac{1}{Re}S_2 S_0^{-1}U = M_1(U) + M_2(U)$$

where $M = S_2 S_0^{-1}$. This constitutes an ordinary differential equation for U with respect to time t .

The second term on the right-hand side represents the nonlinear term of the Burgers equation. To enable integration, linearization is performed using Taylor expansion at $t = t_i$:

$$U(t) \approx U(t_i) + \left. \frac{dU}{dt} \right|_{t_i} (t - t_i)$$

Substituting this into the equation and rearranging yields the time-marching formula:

$$U^{n+1} = T U^n + \tau B^n$$

where the exponential matrix $T = \exp(H\tau)$ can be solved precisely. First, T is expressed as:

$$T = \exp(H\tau) = \left[\exp\left(\frac{H\tau}{m}\right) \right]^m$$

When m is sufficiently large, $\tau = \frac{\Delta t}{m}$ becomes a very small time increment, making τH approach the zero matrix. Expanding $\exp(\tau H)$ via Taylor series about the zero matrix:

$$\exp(\tau H) \approx I + \tau H + \frac{(\tau H)^2}{2!} + \frac{(\tau H)^3}{3!} + \dots$$

Notably, $T_a = \exp(\tau H) - I$ approaches the zero matrix and must be stored separately from the identity matrix I to maintain precision. Through m iterations of:

$$T_a = 2T_a + T_a \cdot T_a$$

the high-precision solution $T = I + T_a$ can be obtained.

1.3 New Method for Time-Stepping with Dynamically Adaptive Grids

In traditional implementations of the dynamically adaptive multiscale wavelet collocation method, to enable numerical computation to track solution singularities, the collocation set retains not only points satisfying the condition $|d_{jk}| > \varepsilon_j$ (where ε_j is a scale-dependent constant and ε is the wavelet coefficient threshold) from the previous time level, but also important points that may contribute to solution singularities at the current time level. Since numerical time steps are small, solution singularities can only migrate from previous collocation points to their immediate neighbors. Therefore, important points that may contribute to singularities at the current time level are located adjacent to points satisfying the threshold condition from the previous time level.

Analysis of the irregular collocation set construction method in traditional dynamically adaptive multiscale wavelet collocation methods, verified through numerical experiments, leads to the following conclusion: when the time step Δt is too large, the traditional method fails to track solution singularities, and its irregular collocation set construction method becomes a primary factor affecting numerical stability.

To enhance numerical stability, this paper improves the dynamic adaptive grid construction approach through a “solve-as-you-generate” method. The implementation steps for the fast dynamically adaptive multiscale wavelet collocation method based on precise time-integration are as follows:

1. Select interpolation wavelets and construct matrices A , B , and S_1 required for time advancement.

2. Assume scaling function or wavelet coefficients c_i^{n-1} are obtained on irregular collocation set G_{i-1} at the previous time level. Compute the approximate solution u_i^{n-1} on collocation set G .
3. On the coarsest-scale scaling space collocation set G_0 within G , compute current time-level coefficients c^n for $G \in G_0$.
4. Assume smaller-scale wavelet space collocation set G_2 within G merges with G_0 and G_1 to form G_2^+ , where collocation points in G_2 interlace with those in G_0 and G_1 as neighboring points. If coefficients at adjacent points satisfy $|c^n| > \varepsilon_j$ for $G \in G_2$, compute coefficients c^n at these points.
5. Similarly compute coefficients on other wavelet space collocation sets, ultimately obtaining coefficients c_i^n on points within G that may contribute to solution singularities.
6. Retain collocation points via thresholding to construct the irregular collocation set G_i at the current time level, compute the approximate solution u_i^n at time t_i on collocation set G .
7. Repeat steps (2)-(6) for time-level advancement until obtaining the final solution u^n at time t .

2 Numerical Experiments

2.1 Problem Description Equation (1) presents the one-dimensional dimensionless Burgers equation. The computational domain is $(0, 2)$ with $Re = 1000$. The initial and boundary conditions are:

$$u(x, 0) = \sin(\pi x), \quad u(0, t) = u(2, t) = 0$$

Both the traditional dynamically adaptive multiscale wavelet collocation method using a first-order explicit temporal scheme and the proposed first-order precise time-integration based method are employed for solution. Symmetric extension is applied at boundaries, and the computational domain is uniformly discretized into $2^{10} + 1$ collocation points.

2.2 Analysis of Results With wavelet threshold $\varepsilon = 10^{-5}$ and $\Delta t = 10^{-4}$, the proposed fast dynamically adaptive multiscale wavelet collocation method based on precise time-integration is applied. [Figure 1: see original paper] presents the numerical results at various dimensionless times along with the dynamically adaptive collocation points used in the computation. The results show that the Burgers equation evolves from a smooth initial field to a solution with strong local singularities. Local singular points in the physical field correspond to high-frequency fluctuations, causing increased wavelet coefficients at fine-scale collocation points. As local singularities intensify, the method adaptively adds fine-scale collocation points, demonstrating its capability to identify singularities.

To compare the stability of both algorithms, numerical simulations are conducted with time steps $\Delta t = 10^{-4}$ and $\Delta t = 10^{-3}$ at wavelet threshold $\varepsilon = 10^{-5}$. [Figure 2: see original paper] compares the computational results and collocation point usage for the one-dimensional Burgers equation using both methods at different time steps. The results demonstrate that the proposed method effectively identifies, captures, and tracks solution singularities at both $\Delta t = 10^{-4}$ and $\Delta t = 10^{-3}$, accurately simulating the physical problem. While the traditional method also achieves accurate simulation at the smaller time step $\Delta t = 10^{-4}$, it exhibits error accumulation and amplification at the larger time step $\Delta t = 10^{-3}$ as solution singularities strengthen, leading to oscillations and numerical divergence at $t = 0.38$. These results confirm that the new algorithm exhibits superior numerical stability due to: (1) the precise time-integration method for time-level advancement, which effectively suppresses error amplification despite both methods having first-order temporal accuracy; and (2) the irregular collocation set construction in the proposed algorithm imposes no time step constraints, unlike the traditional method which requires small time steps to track singularities.

presents the CPU time for both algorithms under various time steps and wavelet thresholds. The traditional method diverges at larger time steps, while at smaller time steps its computational time is significantly higher—approximately 10 times greater than the proposed method under equivalent conditions.

Conclusion

This paper combines the advantages of traditional dynamically adaptive multi-scale collocation methods with precise time-integration algorithms to propose a novel method. The following contributions are made:

1. A new method for constructing irregular collocation sets in dynamic adaptive grids is presented, eliminating time step size constraints on collocation set construction and improving computational stability.
2. Time-marching formulas with scaling function or wavelet coefficients as variables are established.

Numerical experiments demonstrate the method's capability to identify, capture, and track solution singularities. Comparative numerical calculations prove the superiority of the proposed method, manifesting in enhanced numerical stability and computational efficiency—achieving approximately 10 times faster computation under equivalent conditions.

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