On Plane Stress State and Stress Free Deformation of Thick Plate with FGM Interface under Thermal Loading: Postprint

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Abstract

The rainbow schlieren deflectometry has been combined with the computed tomography to obtain three-dimensional density fields of shock containing free jets and we call the method the schlieren CT. Experiments on the schlieren CT have been performed at a nozzle pressure ratio of 4.0 by using an axisymmetric convergent nozzle with an inner diameter of 10 mm at the exit where the nozzle was operated at an underexpanded condition. Multidirectional rainbow schlieren pictures of an underexpanded sonic jet can be acquired by rotating the nozzle about its longitudinal axis in equal angular intervals and the three-dimensional density fields are reconstructed by the schlieren CT. The validity of the schlieren CT is verified by a comparison with the density fields reconstructed by the Abel inversion method. As a result, it is found that excellent quantitative agreement is reached between the three-dimensional jet density fields reconstructed from both methods.

Full Text

Preamble

This paper demonstrates the plane stress state and the stress-free thermo-elastic deformation of FGM thick plates under thermal loading. First, the Sneddon-Lockett theorem on the plane stress state in an isotropic infinite thick plate is generalized for a case of FGM problems in which all thermo-mechanical properties are optional functions of the depth coordinate. The proof is based on application of the Iljushin thermo-elastic potential to a displacement-type system of equations that reduces it to the plane stress state problem. Then, the existence of purely thermal deformation is proved in two ways: first, it is shown that the unique solution fulfills conditions of simultaneous constant temperature and linear gradation of the thermal expansion coefficient; second, the proof

is based directly on a stress-type system of equations which straightforwardly reduces to compatibility equations for purely thermal deformation if only the stress field is homogeneous in the domain and at the boundary. Finally, a couple of examples of application to an engineering problem are presented.

Keywords: thick plate with FGM interface, plane stress state, stress-free deformation

Introduction

Functionally graded materials (FGMs) provide thermal insulation and mechanical toughness at high temperatures by varying the composition of thermal conductivity coefficient, thermal expansion coefficient, and Young' s modulus from the high-temperature side to the low-temperature side continuously and simultaneously, thereby removing the discontinuity of layered plates. These advantages make FGMs applicable in many fields such as high-performance engines for aerospace vehicles, turbine blades, and heat-resisting tools. A general overview of thermal stresses in FGMs comprises the work by Noda [?].

Numerous analytical solutions of thermo-elastic plane or three-dimensional problems of FGMs take advantage of specific power or exponential function approximation methods for multi-layered composite plates, simultaneously limiting their generality and raising the question of how to reduce the problem. One way to attain this may be generalization of the theorem on the plane stress state in an isotropic thermo-elastic thick plate proved by Sneddon and Lockett in [?]. The authors presented a convincing proof for a problem of a semi-infinite thermo-elastic medium bounded by two parallel planes and loaded by an arbitrary temperature field on one surface. The method of solution employed was the double Fourier transforms. The results confirmed solutions of analogous problems, which were inspiration for their work, received earlier by Sternberg and McDowell [?], based on Green' s function, and by Muki [?], who used a method combining the theory of Fourier series and the Hankel transforms of integral order.

Recent achievements concerning application of FGM layers, treated directly as thermal barrier coating or indirectly as interface between coating and substrate, are mainly focused on plates or shells of thin or moderate thickness in which assumptions of plane stress or simplified 3D stress states are natural consequences of Kirchhoff-Love' s or Reissner-Mindlin' s hypotheses used. Contrary to the aforementioned broad stream of papers, the number of works concerning fully 3D problems, like thick plates or semi-space, is rather limited. Hence, let us mention several of them in chronological order: Senthil and Batra [?], Dai et al. [?], Pan and Han [?], Wang et al. [?], Jabbari et al. [?], Yang et al. [?], Kulikov and Plotnikova [?].

Nomenclature

- **Coefficient of thermal expansion**
- **Coefficient of thermal conductivity**
- **Displacement potential**
- **A, B, C**: Poisson' s ratio arbitrary constants
- **Young' s and Kirchhoff' s moduli**
- **_r, _ , _z**: cylindrical components of strain tensor
- **_ij, _ij**: stress tensor: axial and shear components
- **Kronecker' s symbol**
- **Dilatation**
- **Temperature and temperature change**
- **s = tr(_ij)**: trace of stress tensor
- **{u, w}**: axially symmetric displacement vector
- **Displacement vector**
- **Cylindrical co-ordinates**
- **Cartesian co-ordinates**

FGM' s -Concept, Fabrication and Numerical Modeling

In many applications, especially in the space industry as well as the electronic industry, structures or parts of structures are exposed to high temperatures, usually up to 2000K or even 3500K in some parts of rocket engines, high temperature gradients, and/or cyclic temperature changes. Conventional metallic materials, such as carbon steels or stainless steels: ASTM 321, ASTM 310, nickel- or aluminium-based alloys cannot resist such high temperatures. The first method to improve the resistance of metallic structures against extreme temperature conditions consists in covering the structure with a ceramic layer since ceramics are known for their high thermal resistance. For instance, in a metal-ceramic composite Al-SiC, the thermal conductivities ratio is approximately equal: _m/ _c = 3.6, the thermal expansion coefficients ratio: _m/ _c = 5, whereas the elastic moduli ratio: E_m/E_c = 0.16. Indices m and c refer to metallic and ceramic materials respectively. Hence, at the metal-ceramic interface, severe discontinuity of thermo-mechanical properties occurs, which results in high strain and stress mismatch at the interface. As a consequence, delamination or failure of the coating is rapidly observed. As a remedy to these disadvantages, the concept of Functionally Graded Materials (FGM) was developed in Japan in the 1980s, giving structural components a spatial gradient in thermo-mechanical properties. The spatial gradient is achieved by use of two-component composites. The volume fraction of the composite constituents varies spatially such that the effective thermo-mechanical properties change smoothly from one material (ceramic) to the other (metal). In this way, in the case of a Thermal Barrier Coating deposited on a metallic substrate, the heat-resistant ceramic layer and the solid metal are separated by a functionally graded FG layer, the composition of which varies from pure ceramic to pure metal. The processing technologies for TBCs and FGMs may lead to residual stresses, which are built-in during cool-down from the elevated fabrication temperature. These residual stresses may be significant relative to thermo-mechanical stresses applied subsequently. As regards FG layer processing, Plasma Spray Thermal Barrier Coating leads to lamellar microstructures, whereas columnar-lamellar micro-structures are produced when using Electron Beam Physical Vapour Deposition, see Fig. 1 [Figure 1: see original paper].

Fig. 1. Microstructure of chemically graded Electron Beam Physical Vapour Deposition thermal barrier coating, after Schulz et al. [**?**].

When the classical FEM based on homogeneous elements is used for FGMs, the material properties stay the same for all integration points belonging to one finite element. This means that material properties may vary in a piecewise continuous manner, from one element to the other, and a unique possibility to model FGM structure is approximation by use of appropriately fine mesh. On the other hand, a too coarse mesh may lead to unrealistic stresses at the interface between the subsequent layers. To overcome this difficulty, a special graded element has been introduced by Kim and Paulino [**?**] to discretize FGM properties. The material properties at Gauss quadrature points are interpolated there from the nodal material properties by the use of isoparametric interpolation functions. Contrary to the classical FEM formulation, the stiffness matrix of an element is expressed by the integral in which the constitutive matrix is a function of the coordinates. However, from the numerical point of view, nothing stands in the way of implementation of shape functions referring directly to the individual character of inhomogeneity, for instance power functions or exponential functions.

General Formulation of FGM Thermoelastic Problem

A thermo-elastic three-layer body under consideration (Fig. 2 [Figure 2: see original paper]) is bounded by two parallel planes normal to axis x_3, and its FGM interface thermo-mechanical properties such as , , E and G are arbitrary functions of x_3.

Fig. 2. Thick thermoelastic plate with FGM interface under arbitrary thermal load.

The plate is subjected to a temperature field T + (x_i), where T is the temperature of the solid corresponding to zero stress and strain. It is also assumed that there are no body forces within the solid and that its surfaces are free from tractions.

The system of equations of uncoupled thermo-elasticity expressed in displacements takes the form in which Poisson' s ratio is independent of x_3. All terms in Eqs (1) containing partial differentials of , , E and G with respect to x_3 are a consequence of FGM application and they are additional compared with the classical formulation of homogeneous material. The variation of temperature throughout the solid is determined by the steady Fourier equation Eq. (13) in case of absence of inner heat sources. The relation between the stress tensor _ij and the displacement vector u_i is given by the Duhamel-Neumann equation.

The stress components referring to the plane stress state with respect to axis x_3 satisfy as identity the two first equations of the following system as well as the stress components referring to the z-axis when B = 0. Hence, Eqs (10) are reduced to a form analogous to (8) as follows.

System of Equations Expressed in Stresses

The system of equations expressed in stresses (extension of Beltrami-Michell formulation) equivalent to Eq. (1) is as follows:

$$1, 2, \text{no sum over } \alpha, \beta \quad \text{and} \quad \partial_\alpha \partial_\beta$$

It is worth noticing that equations (31-3) can be obtained either in the classical way or directly from equations (11-2) according to the concept by Ignaczak [?].

Conditions of Plane Stress State

To solve Eqs (1), the following potential, originally proposed by Iljushin et al. [?], is introduced:

$$\phi = \text{function of displacement potential}$$

where the function of displacement potential is of harmonic type. Simple introduction of definitions (3) into Eqs (1) shows that only equations of mechanical state are satisfied as identity.

Differentiation of Eq. (121) with respect to r and subsequent substitution, according to Eqs (91), leads to the classical Euler-type differential equation describing the thermo-mechanical membrane state. The unique solution of equation (13) that satisfies boundary conditions takes the form:

$$u(r, z) = \int \theta(\rho)\rho \, d\rho$$

It is obvious that the Iljushin potential (4) rewritten in the form suitable for axial symmetry (x_1 = r, x_3 = z, u_1 = u and u_3 = w) yields:

$$\theta(r, z) = \int \theta(\rho)\rho \, d\rho$$

Additionally, in the case when temperature is bounded, the solution (15) reduces to:

$$u(r, z) = \int \theta(\rho)\rho \, d\rho$$

and it is clear that its dependence with respect to the depth coordinate z comes from the functional gradation of Young' s modulus E(z) and thermal expansion coefficient (z) as well as the temperature field non-homogeneity (r, z) exclusively.

Conditions of Stress-Free Deformation

Constant temperature   = const and linear gradation of coefficient of thermal expansion (z) = _0 + _1z substituted into Eqs (17) leads formally to linear (stress-free) deformation, although constant temperature does not satisfy condition (161).

The proof of the above theorem in the case of the stress formulation Eqs. (31-3) is straightforwardly analogous to those done by Fung [?] and Nowacki [?] for homogeneous material. This turns out to be almost elementary when in both Eqs. (31-3) one assumes that _ij   0 and _ij   0, and appropriate boundary conditions, hence the system of equations is satisfied as identity if:

$$\sigma_{ij} = 0, \quad \tau_{ij} = 0 \quad \text{for } i, j = 1, 2, 3$$

Once again assuming constant temperature   = const which satisfies Fourier' s law (193), the unique solution of (191-2) is the linear gradation of coefficient of thermal expansion (z) = _0 + _1z. It is essential to notice here that uniqueness of solution of the stress formulation (3) requires continuity and smoothness of the stress as well as the thermal strain term   .

Examples of Application

Plane stress state of thick plate made of FGM Al/ZrO +Y O

Let us assume that all thermo-mechanical properties of the three-layer FGM depend on the local magnitude of volume fraction of both constituents which is subjected to the tangent hyperbolic approximation:

$$p(z) = p_c + \frac{p_m - p_c}{2} \left[1 + \tanh\left(\frac{z-a}{b}\right)\right]$$

where p(z) stands for respective property (z), (z) or E(z), indices "c" and "m" refer to ceramic or metallic materials, and parameters a and b define location and thickness of the interface layer.

Differentiation of Eq. (21), division by p(z), and substitution of p_c = _c and p_m = _m allow us to easily find that the coefficient of thermal nonhomogeneity in Eq. (103) equals:

$$\frac{p'(z)}{p(z)} = \frac{1}{b} \text{sech}^2\left(\frac{z-a}{b}\right)$$

Exemplary distributions of   and   are shown in Fig. 4 [Figure 4: see original paper]. Applying the finite difference method, one may solve Eq. (103) according to the scheme shown in Fig. 5 [Figure 5: see original paper], whereas appropriate schemes of boundary conditions allowing for elimination of nodes situated outside the domain are as follows (23).

The finite difference representation allows us to substitute the boundary value problem of partial differential equations by a problem of searching for solution of a system of N linear equations involving N unknowns. This system of equations exhibits typical features of sparse matrix systems having a relatively small number of nonzero elements; hence the natural way to solve it is the application of the row-indexed storage mode [?] combined with the conjugate gradient method [?].

Obviously, only a small fragment neighboring the axis of symmetry of the whole infinite structure is considered. The finite difference operator, shown in Fig. 5, is spanned over a mesh of 161 × 81 square elements with Δr = Δz. The thermal load applied to the upper surface of the plate is subjected to the following relation:

$$\theta(r,0) = \theta_0 e^{-r^2/r_0^2}$$

Temperature distribution is shown in Fig. 6 [Figure 6: see original paper]. In comparison to the temperature distribution obtained for homogeneous material (see Cegielski [?]), the temperature field exhibits a drastic decrease of temperature at the top layer. This is a consequence of application of ceramic material having a coefficient of thermal conductivity 77 times lower than the analogous coefficient of metallic substrate. Hence, one may clearly observe the effect of thermal barrier coating with characteristic strong temperature gradients in it and simultaneous homogenization of the temperature field in middle and bottom layers. This effect is more clearly visible in the case of temperature gradient field presented in Fig. 7 [Figure 7: see original paper]. The biggest magnitudes of temperature gradient, referring to top fibers of the plate, are almost 10 times bigger than analogous values at bottom fibers.

Solution of the mechanical problem is illustrated by distribution of hoop stress, which turns out to be the dominant component of stress, in Fig. 8 [Figure 8: see original paper]. As a consequence, a ceramic material of very low or zero tensile strength is obviously unable to carry tensile stress unless there exists residual stress built into the material, coming from the fabrication process, big enough to neutralize tensile hoop stress. Otherwise, metal-ceramic FGM has to be replaced by metal-metal FGM which exhibits sufficient tensile strength.

Stress-free deformation state of linear FGM interface under constant temperature

It has been shown in the previous section that a material with linear gradation of thermal expansion coefficient, subjected to constant temperature exclusively, is not stressed. This means that it exhibits unconstrained and purely thermal deformation. In the case of axial symmetry, such deformation can be expressed by the following equations:

$$u(r) = \alpha_0 r\theta, \quad w(z) = \alpha_1 z\theta$$

Assuming a structure composed of homogeneous metallic substrate (Al) and ceramic layer (Al O), joined by an FGM interface, shown in Fig. 10 [Figure 10: see original paper], and thermo-elastic properties presented in Table 1, such that the linearly graded coefficient of thermal expansion exhibits a polygonal function, the displacement field corresponding to stress-free deformation defined by Eqs (27-28) is spanned over a mesh of 80×40 square elements and shown in Fig. 11 [Figure 11: see original paper]. It is well visible that both substrate and ceramic layers exhibit homogeneous deformation, whereas deformation of the interface is curvilinear. Although all three deformations satisfy individually stress-free state, they are not compatible since the previously mentioned conditions are violated.

In order to obtain compatible deformation (see Fig. 12 [Figure 12: see original paper]), it is necessary to activate a non-zero stress state whose magnitude can be controlled by the thickness of the interface. The general tendency is as follows: the narrower the thickness of the interface, the lower the magnitude of stress that occurs. Analogously to the temperature field, application of functionally graded composite leads to concentration of compressive stress in the top layer being ceramic material of high toughness. This convenient effect is accompanied by simultaneous unloading of middle and bottom layers built of metallic substrate. Nevertheless, another effect of tensile stress zone in the ceramic layer occurs. This phenomenon is strictly associated with the structure of equations defining stress components (17). Namely, as far as the radial stress is always negative, the hoop stress frequently changes sign, see Fig. 9 [Figure 9: see original paper].

Conclusions

The following concluding remarks may be formulated for thick FGM plates:

- Thermal loading applied to the structure results in the plane stress state if only force-type boundary conditions are homogeneous and there are no body forces.
- There is no need to limit considerations to problems of specific power or exponential approximation functions since after application of Iljushin' s potential only Fourier' s equation turns out to have varying coefficient.
- Application of functionally graded composite Al/ZrO +Y O is very efficient since the FGM layer works like a thermo-mechanical barrier, successively protecting metallic substrate from both high temperature gradients and high concentration of compressive stress.
- Occurrence of tensile hoop stress in the ceramic layer is admissible only when it is accompanied by appropriate compressive residual stress.
- Stress-free deformation of a three-layer structure is not possible because of lack of compatibility; hence probably the unique way to decrease magnitude of stress leads to application of the FGM interface as thin as possible.
- Both theorem and examples of its application have only theoretical sense since neither manufacturing nor classical FEM allow for modeling of continuously varying FGM.

References

[1] Noda, N., 1999, Thermal stresses in functionally graded materials, *J. Thermal Stresses*, 22(4/5), pp. 477 512.

[2] Sneddon, I.N., and Lockett, F.J., 1960, On the steady-state thermoelastic problem for the half-space and the thick plate, *Quart. Appl. Math.*, 18(2), pp. 145-153.

[3] Sternberg, E., and McDowell, E.L., 1957, On the steady-state thermoelastic problem for the half-space, *Quart. Appl. Math.*, 14, p. 381.

[4] Muki, R., 1957, Thermal stresses in a semi-infinite solid and a thick plate under steady distribution of temperature, *Proc. Fac. Eng. Keio. Univ.*, 9, p. 42.

[5] Senthil, S.V., and Batra, R.C., 2003, Three-dimensional analysis of transient thermal stresses in functionally graded plates, *Int. J. Solids Struct.*, 40, pp. 7181 7196.

[6] Dai, K.Y., Liu, G.R., Han, X., and Lim, K.M., 2005, Thermomechanical analysis of functionally graded material (FGM) plates using element-free Galerkin method, *Computer & Structures*, 83, pp. 1487 1502.

[7] Pan, E., and Han, F., 2005, Green' s functions for transversely isotropic piezoelectric functionally graded multi-layered half spaces, *Int. J. Solid Struct.*, 42, pp. 3207

[8] Wang, X., Pan, E., and Roy, A.K., 2007, Three-dimensional Green's functions for a steady point heat source in a functionally graded half-space and some related problems, *Int. J. Eng. Science*, 45, pp. 939 950.

[9] Jabbari, M., Shahryari, E., Haghighat, H., and Eslami, M.R., 2014, An analytical solution for steady state three-dimensional thermoelasticity of functionally graded circular plates due to axisymmetric loads, *Eur. J. Mechanics A/Solids*, 47, pp. 124 142.

[10] Yang, K., Feng, W.-Z., Peng, H.-F., and Lv, J., 2015, A new analytical approach of functionally graded material structures for thermal stress BEM analysis, *Int. Communications Heat Mass Transfer*, 62, pp. 26 32.

[11] Kulikov, G.M., and Plotnikova, S.V., 2015, A sampling method and its implementation for 3D thermal stress analysis of functionally graded plates, *Comp. Struct.*, 120, pp. 315 325.

[12] Schulz, U., Bach, F.W., and Tegeder, G.. 2003, Graded coating for thermal, wear and corrosion barriers, *Mater. Sci. Eng.*, A 362(1 2), pp.61 80.

[13] Kim, J.H., and Paulino, G.H., 2002, Isoparametric graded finite elements for non-homogeneous isotropic and orthotropic materials, *ASME J. Appl. Mech.*, 69, pp. 502 514.

[14] Ignaczak, J., 1959, Direct determination of stresses from the stress equations of motion in elasticity, *Arch. Mech. Stos.*, 11(5), pp. 671 678.

[15] Iljushin, A.A., Lomakin, W.A., and Shmakov, A.P., 1979, *Mechanics of Continuous Media*, Moscow.

[16] Fung, Y.C., 1965, *Foundations of Solid Mechanics*, Prentice-Hall, New Jersey.

[17] Nowacki, W.,1970, *Theory of elasticity*, PWN, Warsaw.

[18] Cegielski, M., 2007, Numerical modeling of combustion engine piston made of MC composite, Diploma Thesis, Cracow University of Technology, Cracow, Poland.

[19] Wang, B.L., Han, J.C., and Du, S.Y., 2000, Crack problems for functionally graded materials under transient thermal loading, *J. Thermal Stresses*, 23, pp. 143 168.

[20] Lee, W.Y., Stinton, D.P., Berndt, C.C., Erdogan, F., Lee, Y.-D., and Mutasin, Z., 1996, Concept of functionally graded materials for advanced thermal barrier coating applications, *J. Am. Ceram. Soc.*, 79, 3003 3012.

[21] PCGPAK User' s Guide, New Haven: Scientific Computing Associates, Inc.

[22] Press, W.H., Teukolsky, S.A., Vetterling, W.T., and Flannery, B.P., 1983, *Numerical Recipes in Fortran*, Cambridge Univ. Press.

Note: Figure translations are in progress. See original paper for figures.