

## Cosmological Scaling Solutions and Cross-coupling Exponential Potential Postprint

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### Full Text

## Cosmological Scaling Solutions and Cross-coupling Exponential Potential

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### Abstract

We present a phase-space analysis of cosmology containing multiple scalar fields with a positive or negative cross-coupling exponential potential. We show that there exist power-law kinetic-potential-scaling solutions for a sufficiently flat positive potential or for a steep negative potential. The former is the unique late-time attractor, but it is difficult to yield assisted inflation. The latter is never stable in an expanding universe. Moreover, for a steep negative potential

there exists a kinetic-dominated regime in which each solution is a late-time attractor. In the presence of ordinary matter these scaling solutions with a negative cross-coupling potential are found unstable. We briefly discuss the physical consequences of these results.

## 1 Introduction

Scalar field cosmological models are of great importance in modern cosmology. The dark energy is attributed to the dynamics of a scalar field, which convincingly realizes the goal of explaining the current accelerating expansion of the universe generically using only attractor solutions. Furthermore, a scalar field can drive an accelerated expansion and thus provides possible models for cosmological inflation in the early universe. In particular, there have been a number of studies of spatially homogeneous scalar field cosmological models with an exponential potential. They are already known to have interesting properties; for example, if one has a universe containing a perfect fluid and such a scalar field, then for a wide range of parameters the scalar field mimics the perfect fluid, adopting its equation of state.

These scaling solutions are attractors at late times. The inflation models and other cosmological consequences of multiple scalar fields have also been considered.

The scale-invariant form makes the exponential potential particularly simple to study analytically. There are well-known exact solutions corresponding to power-law solutions for the cosmological scale factor  $a \propto t^p$  in a spatially flat Friedmann-Robertson-Walker (FRW) model. More generally, the coupled Einstein-Klein-Gordon equations for a single field can be reduced to a one-dimensional system which makes it particularly suitable for a qualitative analysis. Recently, adopting a system of dimensionless dynamical variables, the cosmological scaling solutions with positive or negative exponentials have been studied. In general, there are many scalar fields with exponential potentials in supergravity, superstring and the generalized Einstein theories, thus multiple potentials may be more interesting. In the previous paper, we studied the stability of cosmological scaling solutions in an expanding universe model with multiple scalar fields with positive or negative exponential potentials. A phase-space analysis of the spatially flat FRW models shows that there exist cosmological scaling solutions which are the unique late-time attractors and successful inflationary solutions driven by multiple scalar fields with a wide range of each potential slope parameter  $n$ . It is assumed that there is no direct coupling between potentials. Multiple cross-coupling exponential potentials arise on many occasions, for instance, from compactifications of vacuum Einstein gravity on product spaces. Indeed, they are a natural outcome of the compactification of higher dimensional theories down to 3+1 dimensions. With this in mind, it is worth investigating such potential in a bit more detail.

In this paper, we first study a system of dimensionless dynamical variables

of multiple scalar fields with a positive or negative cross-coupling exponential potential. We obtain the scaling solutions and analyze their stability. There still exist cosmological scaling solutions which are the unique late-time attractors. In this model we then introduce a barotropic fluid to the system. We discuss the physical consequences of these results.

## 2 Cross-coupling Exponential Potential

We consider  $n$  scalar fields  $\phi_i$  with a cross-coupling potential  $V = V_0 \exp(-\sum_i \alpha_i \phi_i)$  where  $\alpha_i^2 \leq 8 G_N$  is the gravitational coupling and  $\alpha_i$  are dimensionless constants characterizing the slope of the potential. Further, we assume all  $\alpha_i \geq 0$  since we can make them positive through  $\phi_i \rightarrow -\phi_i$  if some of them are negative. The evolution equation of each scalar field for a spatially flat FRW model with Hubble parameter  $H$  is:

$$\ddot{\phi}_i + 3H \dot{\phi}_i - \alpha_i^2 V = 0$$

subject to the Friedmann constraint:

$$H^2 = \frac{1}{3} \sum_i \dot{\phi}_i^2 + V$$

Defining  $(n+1)$  dimensionless variables:

$$x_i = \dot{\phi}_i / (\sqrt{6} H) \quad y = \sqrt{V} / (\sqrt{6} H)$$

the evolution equations can be written as an autonomous system:

$$x'_i = -3x_i + \alpha_i \sqrt{6} y^2 / 2 \quad y' = -y \sum_j x_j$$

where a prime denotes a derivative with respect to the logarithm of the scale factor,  $N = \ln a$ , and the constraint equation becomes:

$$\sum x_i^2 + (\pm) y^2 = 1$$

Throughout this paper we will use upper/lower signs to denote the two distinct cases of  $(\pm) V > 0$ .  $x_i^2$  measures the contribution to the expansion due to the field's kinetic energy density, while  $(\pm) y^2$  represents the contribution of the potential energy. We will restrict our discussion of the existence and stability of critical points to expanding universes with  $H > 0$ , i.e.,  $y > 0$ . Critical points correspond to fixed points where  $x'_i = 0$  and  $y' = 0$ , and there are self-similar solutions with  $H^2 = a^{-p}$ . This corresponds to an expanding universe with a scale factor  $a(t)$  given by  $a \propto t^{\hat{p}}$ , where  $p = 1/3$ .

The system has at most one  $n$ -dimensional sphere  $S$  embedded in  $(n+1)$ -dimensional phase-space corresponding to kinetic-dominated solutions, and a fixed point  $A$ , which is a kinetic-potential-scaling solution listed in Table 1.

In order to study the stability of the critical points, using the Friedmann constraint equation we first reduce the equations to  $n$  independent equations:

$$x'_i = -3x_i + \alpha_i \sqrt{6} (1 - \sum x_j^2) / 2$$

Substituting linear perturbations  $x_i \rightarrow x_i + \delta x_i$  about the critical points into the equations, to first-order in the perturbations, gives equations of motion:

$$\delta x_i' = -2\sqrt{6} \delta x_i y - 3 \delta x_i + 3x_i \Sigma (2x_j - x_j) \quad y' = -y \Sigma - \Sigma x_j - (\Sigma x_j) y$$

which yield  $n$  eigenvalues  $m_i$ . Stability requires the real part of all eigenvalues being negative.

**S:**  $\Sigma x_i^2 = 1, y = 0$ . These kinetic-dominated solutions always exist for any form of the potential, which are equivalent to stiff-fluid dominated evolution with a  $t^{1/3}$  irrespective of the nature of the potential. Then the perturbation equations become:

$$\delta x_i' = -3 \delta x_i + 3x_i \Sigma (2x_j - x_j)$$

which yield  $n$  eigenvalues: one of them, say  $m$ , does not vanish,  $m = -3(1 - \Sigma x_j)$ , while the remainder of them vanish. Thus the solutions are marginally stable for  $\Sigma x_i > 0$  and unstable for  $\Sigma x_i < 0$ .

**A:**  $x_i = \delta_i / \sqrt{(\Sigma \delta_j^2)}, y = \sqrt{((1 - \Sigma \delta_i^2)/6)} / (\pm)$ . The potential-kinetic-scaling solution exists for  $\Sigma \delta_i^2 < 6$  positive potentials or steep negative potentials. The power-law exponent,  $p = \Sigma \delta_i^2 / (6 - \Sigma \delta_i^2)$ , depends on parameter  $\delta_i$ . From the perturbation equations we find the eigenvalues:

$$m_i = -3(1 - \delta_i^2 / \Sigma \delta_j^2)$$

Thus the scaling solution is always stable when this point exists for a positive potential, which corresponds to power-law inflation in an expanding universe when  $\Sigma \delta_i^2 < 2$ . However, this solution is unstable for a negative potential.

The different regions of  $\delta_i$  parameter space lead to different qualitative evolution. As an example we consider the cosmologies containing  $n$  scalar fields with the cross-coupling potential with  $\delta_i = \delta$ . For the sufficiently flat ( $\delta^2 < 6/n$ ) positive potential, these kinetic-dominated solutions are unstable and the kinetic-potential-scaling solution is the stable late-time attractor. Hence generic solutions start in the former and approach the latter at late times. For the steep ( $\delta^2 > 6/n$ ) positive potential, there exists a stable kinetic-dominated regime, in which each point is the late-time attractor. Hence generic solutions start in kinetic-dominated solution and approach the stable regime. For the sufficiently flat ( $\delta^2 < 6/n$ ) negative potential, only these kinetic-dominated solutions exist which are unstable scaling solutions. For the steep ( $\delta^2 > 6/n$ ) negative potential, the kinetic-potential-scaling solution is unstable and there exists a stable kinetic-dominated regime. Hence generic solutions start in a kinetic-dominated regime or the kinetic-potential-scaling solution and approach the stable kinetic-dominated regime at late times.

### 3 Plus a Barotropic Fluid

We now consider multiple scalar fields with the cross-coupling potential evolving in a spatially flat FRW universe containing a fluid with barotropic equation of state  $P_- = (\gamma - 1)\rho_-$ , where  $\gamma$  is a constant,  $0 < \gamma < 2$ , such as radiation ( $\gamma = 4/3$ ) or dust ( $\gamma = 1$ ).

The evolution equation for the barotropic fluid is:

$$\dot{\rho}_- = -3H(\rho_- + P_-)$$

subject to the Friedmann constraint:

$$H^2 = \frac{1}{2} \sum \dot{x}_i^2 + V + \rho_-$$

We define another dimensionless variable:

$$z = \sqrt{\rho_-} / (\sqrt{6} H)$$

The evolution equations can then be written as an autonomous system:

$$\begin{aligned} x'_i &= -3x_i + \dot{x}_i \sqrt{6} y^2 / 2 - (3/2) x_i z^2 y' = -y \sum_j x_j + (3/2) y z^2 \\ z' &= -(3/2) z (1 - z^2) + (3/2) z \sum x_j^2 \end{aligned}$$

and the constraint equation becomes:

$$\sum x_i^2 + (\pm) y^2 + z^2 = 1$$

Critical points correspond to fixed points where  $x'_i = 0$ ,  $y' = 0$  and  $z' = 0$ , and there are self-similar solutions with  $H^2 = a^{-p}$ . This corresponds to an expanding universe with a scale factor  $a(t)$  given by  $a \propto t^{\hat{p}}$ , where  $\hat{p} = 2/(3 - p)$ .

The system has at most one  $n$ -dimensional sphere  $S$  embedded in  $(n+2)$ -dimensional phase-space corresponding to kinetic-dominated solutions, a fixed point  $A$  which is a kinetic-potential-scaling solution, a fixed point  $B$  which is a fluid-dominated solution, and a fixed point  $C$  which is a fluid-potential-kinetic-scaling solution listed in Table 2.

**S:**  $\sum x_i^2 = 1$ ,  $y = 0$ ,  $z = 0$ . These kinetic-dominated solutions always exist for any form of the potential, which are equivalent to stiff-fluid dominated evolution with  $a \propto t^{1/3}$  irrespective of the nature of the potential. The linearization of the system about these fixed points yields:

$$x'_i = -3x_i + 3x_i \sum_j (2x_j - x_j) z' = (3/2 - 3)z$$

which indicate that the solutions are marginally stable for  $\gamma = 2$ .

**A:**  $x_i = \dot{x}_i / \sqrt{(\sum \dot{x}_j^2)}$ ,  $y = \sqrt{((1 - \sum \dot{x}_i^2/6) / (\pm))}$ ,  $z = 0$ . The potential-kinetic-scaling solution exists for sufficiently flat potentials. The power-law exponent,  $p = \sum \dot{x}_i^2 / (6 - \sum \dot{x}_i^2)$ , depends on the slope of the potential. The linearization of the system about this critical point yields  $(n+1)$  eigenvalues:

$$m_i = -3(1 - \dot{x}_i^2 / \sum \dot{x}_j^2) \quad m_z = (3/2) - 3$$

Thus the scaling solution is stable for a positive potential with  $\Sigma_{-i}^2 < 3$ , which corresponds to power-law inflation in an expanding universe when  $\Sigma_{-i}^2 < 2$ .

**B:**  $x_{-i} = 0, y = 0, z = 1$ . The fluid-dominated solution exists for any form of the potential, corresponding to a power-law solution with  $p = 2/(3)$ . The perturbation equations indicate that the solution is never stable.

**C:**  $x_{-i} = \frac{-i(2-)}{(\Sigma_{-j}^2)}$ ,  $y = \sqrt{\frac{(2-)}{(\Sigma_{-j}^2 (\pm))}}$ ,  $z = \sqrt{\frac{(\Sigma_{-j}^2 - 3)}{(\Sigma_{-j}^2)}}$ . The fluid-potential-kinetic-scaling solution exists for a positive potential with  $\Sigma_{-i}^2 > 3$ . The power-law exponent,  $p = 2/(3)$ , is identical to that of the fluid-dominated solution, depends only on the barotropic index and is independent of the slope  $-i$  of the potential. The linearization of the system about the fixed point yields  $(n+1)$  eigenvalues:

$$\begin{aligned} m_{-1} &= -3(2-)/\Sigma_{-j}^2 & m_{-2} &= -3(2-)/2 + (3/2)\sqrt{((2-)^2 - 8(\Sigma_{-j}^2 - 3))/\Sigma_{-j}^2} \\ m_{-3} &= \dots = m_{\{n+1\}} = -(2-) \end{aligned}$$

Thus the scaling solution is stable for a positive potential with  $\Sigma_{-i}^2 > 3$ .

The different regions in the  $(\Sigma_{-i}^2, -i)$  parameter space lead to different qualitative evolution. For the sufficiently flat ( $\Sigma_{-i}^2 < 3$ ) positive potential, S, A and B exist. Point A is the stable late-time attractor. Hence generic solutions begin in a kinetic-dominated regime or at the fluid-dominated solution and approach the kinetic-potential-scaling solution at late times. For the intermediate ( $3 < \Sigma_{-i}^2 < 6$ ) positive potential, all critical points exist. Point C is the stable late-time attractor. Hence generic solutions start in a kinetic-dominated regime, at the kinetic-potential-scaling solution or at the fluid-dominated solution and approach the stable fluid-kinetic-potential-scaling solution. For the steep ( $\Sigma_{-i}^2 > 6$ ) positive potential, S, B and C exist. Point C is the stable late-time attractor. Hence generic solutions start in a kinetic-dominated regime or at the fluid-dominated solution and approach the stable fluid-kinetic-potential-scaling solution. For the sufficiently flat ( $\Sigma_{-i}^2 < 3$ ) negative potential, the kinetic-dominated solution S and the fluid-dominated solution B exist, which are unstable. For the intermediate ( $3 < \Sigma_{-i}^2 < 6$ ) negative potential, the kinetic-dominated solution S and the fluid-dominated solution B exist, which are unstable. For the steep ( $\Sigma_{-i}^2 > 6$ ) negative potential, S, A and B exist. Point A is the stable late-time attractor. Hence generic solutions start in a kinetic-dominated regime or at the fluid-dominated solution and approach the stable kinetic-potential-scaling solution at late times.

## 4 Conclusions and Discussions

We have presented a phase-space analysis of the evolution for a spatially flat FRW universe containing  $n$  scalar fields with a positive or negative cross-coupling exponential potential. In particular, for the  $-i =$  case, we find that in the expanding universe model with a sufficiently flat ( $\Sigma_{-i}^2 < 6/n$ ) positive cross-coupling potential the only power-law kinetic-potential-scaling solution

is the late-time attractor. It is more difficult to obtain assisted inflation in such models since the fields with cross-coupling exponential potential tend to conspire to act against one another rather than assist each other. However, steep ( $\Sigma \bar{\rho} > 6/n$ ) negative cross-coupling potential has kinetic-dominated solutions with a  $\bar{\rho} \propto t^{-3}$ , some of which are the late-time attractors. It can be known that the kinetic energy of each field tends to be equal via their effect on the expansion at late times.

Then we have extended the phase-space analysis of the evolution to a realistic universe model with a barotropic fluid plus  $n$  scalar fields with a positive or negative cross-coupling exponential potential. We have shown that for the sufficiently flat ( $\Sigma \bar{\rho} < 3$ ) positive cross-coupling potential, the kinetic-potential-scaling solution is the stable late-time attractor. The energy density of the scalar fields dominates at late times. Moreover, for the steep ( $\Sigma \bar{\rho} > 6$ ) positive cross-coupling potential, the fluid-kinetic-potential-scaling solution is the stable late-time attractor. However, a negative cross-coupling potential has no stable scaling solutions.

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