Off-center moments set of Vlasov-Maxwell system

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Abstract

We propose a more effective fluid description on Vlasov-Maxwell (V-M) system. It is via an open set of off-center moments, which obey an open set of motion equations.This new description is of more advantage to give exact macroscopic information of the V-M system than the well-known moments-description. The new description implies that obtaining exact solutions of all moments is not necessary condition of obtaining those of self-consistent fields.

We propose a more effective fluid description for the Vlasov-Maxwell system. This description employs an open set of off-center moments, which obey an open set of fluid equations. Compared with the well-known moment description, this new description offers greater advantages in obtaining exact macroscopic information of the V-M system. This new description implies that obtaining exact solutions for all moments is not a necessary condition for obtaining those of the self-consistent fields.

Full Text

Off-Center Moments of the Vlasov-Maxwell System

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We propose a more effective fluid description of the Vlasov-Maxwell (V-M) system based on an open set of off-center moments that obey an open set of evolution equations. This new description provides more advantageous access to exact macroscopic information of the V-M system compared to the conventional moment description. Importantly, it implies that obtaining exact solutions for all moments is not a necessary condition for determining exact solutions of the self-consistent fields.

The fluid description of a Vlasov-Maxwell (V-M) system [1] is typically formulated through an open set of moments $\{M_i; 0 \le i < \infty\}$, where $M_i = \int \mathbf{v}^i f d^3\mathbf{v}$, and these moments obey an open hierarchy of equations. Here, $\hat{L}f = 0$ is the Vlasov equation (VE), with the moment equations given by $\int \mathbf{v}^i \hat{L} f d^3\mathbf{v} = 0$ for $0 \le i < \infty$ [2]. The Vlasov operator is $\hat{L} = [\partial_t + \mathbf{v} \cdot \nabla - \mathbf{F}_L(\mathbf{v}) \cdot \partial_{\mathbf{p}}]$, where $\mathbf{F}_L(\mathbf{v}) = e[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)]$ represents the Lorentz force, $\mathbf{p}(\mathbf{v}) = \mathbf{v}\Gamma(\mathbf{v})$, and $\Gamma(\mathbf{v}) = 1 - \mathbf{v} \cdot \mathbf{v}$. This open hierarchy reflects relations among all moments. On the other hand, Maxwell' s equations (MEs) establish a relationship between the self-consistent fields $(\mathbf{E}, \mathbf{B})$ and only the lowest two moments (M_0, M_1). This structure suggests that $(\mathbf{E}, \mathbf{B})$ would depend on all moments, making exact solution seemingly impossible due to the infinite chain of dependencies. This raises a fundamental question: are exact solutions for all moments truly necessary for obtaining exact solutions of $(\mathbf{E}, \mathbf{B})$?

We can define an alternative open set $\{D_i; 0 \le i < \infty\}$ of off-center moments derived from the M-set. Clearly, $D_0 = 0$ and $D_1 = 0$ identically. According to Maxwell' s equations, $(\mathbf{E}, \mathbf{B})$ depend only on (M_0, M_1) and are independent of the D-set. However, the D-set is governed by $\int \mathbf{v}^i \hat{L} f d^3\mathbf{v} = 0$ for $1 \le i < \infty$, where each equation can be expressed in terms of the D-set as:

$$A_i \partial_t D_i + B_{i+1} \nabla D_{i+1} + \sum_{m \ge i+1} C_m D_m = 0,$$

with coefficients A_i, B_i, C_i being known functionals of $\mathbf{E}, \mathbf{B}, M_0, M_1$. Starting from the $i = 1$ case, we can formally express D_2 in terms of all $D_{i \ge 3}$. Substituting this into the $i = 2$ case yields an expression for D_3 in terms of $D_{i \ge 4}$, and so on. Ultimately, we find that all $D_{i \ge 2}$ are determined by D_∞ and the coefficients A_i, B_i, C_i. Thus, the open equation set $\int \mathbf{v}^i \hat{L} f d^3\mathbf{v} = 0$ for $1 \le i < \infty$ does not impose substantial constraints on $\mathbf{E}, \mathbf{B}, M_0, M_1$.

The introduction of the D-set resolves a practical obstacle in obtaining exact solutions for $(\mathbf{E}, \mathbf{B})$. Although $(\mathbf{E}, \mathbf{B})$ are affected by each M_i, only the (M_0, M_1)-dependent portion of each moment contributes substantially to the fields. Each M_i contains a part D_i that has no contribution to $(\mathbf{E}, \mathbf{B})$. The open equation set $\int \mathbf{v}^i \hat{L} f d^3\mathbf{v} = 0$ for $1 \le i < \infty$ reveals relations among these D_i components. In short, exact solutions for the D-set are not a necessary condition for determining $(\mathbf{E}, \mathbf{B})$.

For any V-M system, the following theorem holds:

Theorem: For the Vlasov equation $\hat{L}f = 0$, where we define the decomposition $f(\mathbf{v}) \equiv n_0 * \delta(\mathbf{v} - \mathbf{u})$ with $n_0 = \int f d^3\mathbf{v}$ and $\mathbf{u} = \int \mathbf{v} f d^3\mathbf{v} / \int f d^3\mathbf{v}$, the following statements are true: 1. $\partial_t n_0 + \mathbf{u} \cdot \nabla n_0 = 0$ 2. $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{e}{m}[\mathbf{E} + \mathbf{u} \times \mathbf{B}]$ 3. The decomposition always exists (since $f \ge 0$)

Proof: Based on the above definitions, we note the natural existence of $\mathbf{u} = \int \mathbf{v} f d^3\mathbf{v} / \int f d^3\mathbf{v}$ and $n_0 = \int f d^3\mathbf{v}$. Applying the Vlasov operator to the decomposition:

$$\hat{L}f = [\partial_t + \mathbf{v}\cdot\nabla]n_0 * \delta(\mathbf{v}-\mathbf{u}) + n_0 * [\partial_t + \mathbf{v}\cdot\nabla - \mathbf{F}_L(\mathbf{v})\cdot\partial_{\mathbf{p}}]\delta(\mathbf{v}-\mathbf{u})$$
$$= [\partial_t + \mathbf{u}\cdot\nabla]n_0 * \delta - n_0 * [\partial_t\mathbf{u} + \mathbf{v}\cdot\nabla\mathbf{u} + \mathbf{F}_L(\mathbf{v})\cdot\partial_{\mathbf{p}(\mathbf{v})}] * \delta',$$

where we have used properties of the Dirac delta function: $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$. Rearranging the term $n_0 * \mathbf{v}\cdot\nabla\mathbf{u} * \delta'$, which equals $\mathbf{v}\cdot\nabla\mathbf{u}\partial_{\mathbf{v}}F$, from the right-hand side to the left-hand side, we obtain:

$$\hat{L}f + \mathbf{v}\cdot\nabla\mathbf{u}\cdot\partial_{\mathbf{v}}F = [\partial_t + \mathbf{u}\cdot\nabla]n_0 * \delta - n_0 * [\partial_t\mathbf{u} + \mathbf{F}_L(\mathbf{v})\cdot\partial_{\mathbf{p}(\mathbf{v})}] * \delta'$$
$$= [\partial_t + \mathbf{u}\cdot\nabla]n_0 * \delta - n_0 * [\partial_t\mathbf{u} + (\mathbf{F}_L(\mathbf{v})\cdot\partial_{\mathbf{p}(\mathbf{v})})|_{\mathbf{v}=\mathbf{u}}] * \delta',$$

where we again used $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$. For simplicity, we denote $\hat{L}f + \mathbf{v}\cdot\nabla\mathbf{u}\cdot\partial_{\mathbf{v}}F$ as Ω. It is then straightforward to verify the relations:

$$[\partial_t + \mathbf{u}\cdot\nabla]n_0 = \int \Omega d^3\mathbf{v}$$

$$n_0 * [\partial_t\mathbf{u} + (\mathbf{F}_L(\mathbf{v})\cdot\partial_{\mathbf{p}(\mathbf{v})})|_{\mathbf{v}=\mathbf{u}}] = -\int(\mathbf{v}-\mathbf{u}) * \Omega d^3\mathbf{v}.$$

Clearly, the condition $\Omega = 0$ leads to: 1. $0 = \int \Omega d^3\mathbf{v} \Rightarrow \partial_t n_0 + \mathbf{u}\cdot\nabla n_0 = 0$ 2. $0 = \int(\mathbf{v}-\mathbf{u}) * \Omega d^3\mathbf{v} \Rightarrow \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{e}{m}[\mathbf{E} + \mathbf{u}\times\mathbf{B}]$

The theorem is thus strictly proven.

References [1] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics*, New York: McGraw-Hill, 1977.
[2] A. Vlasov, J. Phys. U.S.S.R 10, 25 (1945).

Note: Figure translations are in progress. See original paper for figures.