Analytical solutions to single scattering of SH waves by a cylindrical fiber with radially gradient interphases

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Abstract

In this work, analytical solutions to the single scattering of SH waves by a cylindrical fiber with two specific radially gradient interphase layers are supported based on the method proposed by P.A. Martin (Martin 2002, JASA). In the first case, 1) shear modulus (r)=e^2 rand the square of wave number, k^2 , is a linear function of 1/r, and 2) (r)=e$^{(- r}$2) and k^2 is a linear function of r^2. For example, analytical solutions to single scattering of SH waves by a Sic fiber with the above two interphase layers embedded in aluminum are presented. The calculated scattering cross sections are compared with values obtained from an approximate method (dividing the continuous varying layer into multiple homogeneous sub-layers). The two methods yield similar results. The contribution of this work benefits the validation of various numerical methods used in the inhomogeneous media.

Full Text

Preamble

Analytical Solutions for Single Scattering of SH Waves by a Cylindrical Fiber with Radially Gradient Interphases

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Abstract: This work presents analytical solutions for the single scattering of SH waves by a cylindrical fiber with two specific radially gradient interphase layers, building upon the method proposed by P.A. Martin (Martin 2002, JASA). In the first case, 1) the shear modulus (cid:2020)(r) = (cid:1857)(cid:2870)(cid:3081)(cid:3045) and the square of the wave number, (cid:1863)(cid:2870), is a linear function of 1/(cid:1870), and 2) (cid:2020)(r) = (cid:1857)(cid:2879)(cid:3081)(cid:3045)(cid:3118) and (cid:1863)(cid:2870) is a linear function of (cid:1870)(cid:2870). As an illustration, analytical solutions for the single scattering of SH waves by a SiC fiber with the aforementioned interphase layers embedded in aluminum are presented. The calculated scattering cross sections are compared with values obtained from an approximate method (dividing the continuously varying layer into multiple homogeneous sub-layers). The two methods yield similar results. The contribution of this work benefits the validation of various numerical methods used for inhomogeneous media.

Keywords: Single scattering; SH wave; Radially gradient interphase

1. Introduction

Multiple scattering of waves occurs throughout our daily lives—the scattering of sound by water droplets in fog, for example, and the scattering of light by fine particles in the air. Research on multiple scattering of waves has a lengthy history by virtue of its ubiquity [1]. The multiple scattering of elastic waves in composites remains an active research topic, as these materials are relatively new. Due to multiple scattering effects, even if the composite' s matrix and inclusions are both elastic, elastic waves propagating in the material become attenuated; this is called attenuation caused by scattering. To date, many theoretical models have been proposed to evaluate this attenuation, as accurate evaluation facilitates the dynamic characterization and non-destructive assessment [2] of composite materials.

Existing research can be roughly divided into two groups. The first group consists of methods based on wave function expansion and configurational averaging techniques, pioneered by Foldy [3-7]. In these methods, the exciting and scattered waves of each inclusion in a composite material are expressed as series of wave functions. The expansion coefficients of the scattered waves for each inclusion are related to the corresponding values of the exciting wave through a matrix T, the components of which are determined by the boundary conditions of the corresponding single scattering problem [8]. For SH waves, T becomes a scalar since there is no mode conversion. After a sequence of mathematical operations, a homogeneous system for the ensemble-averaged expansion coefficients of the exciting waves or scattered waves can be obtained. Solving this homogeneous system yields the attenuation coefficient.

The second group consists of studies on the self-consistent method [9-12], where it is assumed that each inclusion behaves as an isolated one in a medium with the effective properties of the composite. The exciting wave acting on each inclusion is the coherent wave. The attenuation coefficient is then obtained through various consistency conditions; for example, the mean wave field equals the coherent wave field [12] or the total scattering by the inclusions embedded in the effective medium effectively vanishes. It is clear that both of the aforementioned groups of theoretical models necessitate solving a corresponding single scattering problem [8], that is, the scattering of a plane incident wave by a single inclusion embedded in an infinite matrix. The single scattering problem is solved to obtain the far-field scattering amplitude, (cid:1858)( ), in models proposed by Waterman and Truell [3,13] for example, and is solved to obtain the scattering cross section. In most existing theoretical models, the fibers/particulates are usually assumed to be perfectly bonded to the matrix while the interphase layer between the fibers and matrix is neglected. The interphase layer should be accounted for, however, as it features gradient-changed properties that are generated via chemical reaction and atom diffusion during the manufacturing process (or created artificially to improve compatibility between the fibers and the matrix).

To date, the effect of the functional gradient interphase layer on the attenuation of elastic waves in composite materials has been sparsely considered in several works [14-18]. In these previous studies, in order to solve the corresponding single scattering problem, the inhomogeneous interphase layer is usually divided into multiple layers with homogeneous properties to approximate a continuously varying one. The wave fields in the fiber, intermediate sub-layers, and matrix are then expressed as series of wave functions. The continuity conditions of displacements and stresses at all interfaces are listed and the resulting coupled linear equations for the expansion coefficients are solved. When the intermediate sub-layers are sufficiently thin, this yields accurate solutions [15]. This treatment of the gradient interphase layer is straightforward and can be used to deal with any type of radially gradient profile, but the coefficient matrix of the resulting linear system is sometimes ill-conditioned. Additionally, for composites with a high contrast of properties between fibers and the matrix, the interphase layer must be divided into a large number of sub-layers in order to obtain convergent results. This increases the computational cost as well as the condition number of the coefficient matrix [19].

The transfer matrix method is another candidate for the above interphase problem [18]. In this method, the displacements and stresses at the inner surface of each intermediate sub-layer are related to the corresponding values on the outer interface through a transfer matrix M, the components of which are functions of the material properties and geometry uniquely defined by the sub-layer. Using the transfer matrix for each intermediate sub-layer, the displacements and stresses on the outer surface of the whole interphase layer are related to the inner surface. Assuming the fiber, interphase layer, and matrix are perfectly bonded, a linear system for the expansion coefficients of waves in the fiber and matrix is established and solved. For SH waves, the transfer matrix M is a matrix of size 2×2. Therefore, the final resulting linear system to be solved is of size 2, which is much smaller than that in the method described above. Similarly, the final coefficient matrix of the resulting linear system can be ill-conditioned if the sub-layers are thin.

Though several approximate methods to solve the single scattering problem with a radially gradient interphase layer have been proposed, to date, analytical solutions remain elusive. In this work, analytical solutions for single scattering with a radially gradient interphase layer of two specific profiles are reported for SH waves. The derivation process follows the work of P.A. Martin [20], in which general solutions for the single scattering of acoustic waves by an inhomogeneous sphere with spherically symmetrical properties were investigated. The two transformations used in the current derivation process differ from Martin' s due to the different governing equations for distinct waves. We also demonstrate how to extend the proposed transformations to other waves.

The remainder of this paper is organized as follows: Section 2 derives the governing equations for SH waves in a radially gradient medium. General solutions for two specific radially gradient materials are then derived in Section 3. Section 4 presents analytical solutions for the single scattering problem of SH waves by a cylindrical fiber with the two specific interphase layers and a detailed example. Section 5 provides a brief summary and conclusion.

2. Governing Equations

For SH waves, only the out-of-plane displacement has value and can be expressed as (cid:1875) = (cid:1875)((cid:1824), t), where (cid:1824) = ((cid:1876), (cid:1877)) is the position vector and t represents the time variable. The corresponding stress components can be expressed as follows:

where (cid:2020)((cid:1824)) is the shear modulus, which is a function of position. Substitution of the stress components into the governing equations of elastodynamics yields the following equation:

where ((cid:1824)) is the mass density. After a simple mathematical operation, Eq. (2) can be re-written as follows:

where and (cid:2870) are the 2D gradient and Laplace operator, respectively. (cid:1855)((cid:1824)) = (cid:3493) ((cid:1824))/ ((cid:1824)) is the wave speed, which is also a function of position.

Define (cid:1875)((cid:1824), t) = (cid:2020)(cid:3090)((cid:1824))(cid:1873)((cid:1824), t) [20], where is a constant which is determined to force (cid:1873)((cid:1824), t) into the following form:

For SH waves, can be set to be $-1/2$ and then . The values of should be changed for other scalar waves. In this work, only radial inhomogeneity has been considered; that is, (cid:2020)((cid:1824)) and ((cid:1824)) are functions of only the radial coordinate r, so (cid:1837) can be simplified as follows:

hence u satisfies the following equation:

For time-harmonic problems, (,) where (cid:1863)((cid:1870)) = (cid:2033)/(cid:1855)((cid:1870)). Next, we seek the solution to Eq. (6) in the following form:

where n is an integer and (cid:1851)(cid:3041)((cid:2016)) is a cylindrical harmonic function, such as the Legendre function (cid:1842)(cid:3041)(cos).

We have the following relation:

because (cid:1873)(cid:3041) is a function of r and (cid:1851)(cid:3041) is a function of variable (cid:2016), so (cid:1870)(cid:3041)(cid:1851)(cid:3041)((cid:2016)) is a separated solution to the Laplace equation in the polar coordinate system. To this effect:

By substituting Eqs. (7) and (9) into Eq. (6), it is clear that (cid:1873)(cid:3041)((cid:1870)) should satisfy the following equation:

which is a second-order differential equation for (cid:1873)(cid:3041)((cid:1870)).

3. Two Cases of Radially Gradient Interphase Layers

Two kinds of radially gradient interphase layers are considered here.

Case 1: In this case, the shear modulus is an exponential function of r and (cid:1863)(cid:2870)((cid:1870)) is a linear function of (cid:1870)(cid:2879)(cid:2869) as follows:

where (cid:2020)(cid:2869), , (cid:1863)(cid:2869), (cid:2870), and are constants that are specified according to the following relation:

where (cid:2020)(cid:3033), (cid:2020)(cid:3040) and (cid:1863)(cid:3033), (cid:1863)(cid:3040) are the shear modulus and wavenumber of the fibers and matrix, respectively; a and b are the inner and outer radius of the interphase layer. After a mathematical operation, the expression for (cid:1837)((cid:1870)) in this case can be expressed as follows:

Then the corresponding Eq. (10) becomes:

In order to solve this equation, we make the substitution:

with

where c and d are constants. Here, c is set to be −(cid:2869) and d=1. The general process to specify these two constants is provided in the appendix. is a parameter that can be selected at the operator' s disposal. Eq. (14) can be transformed as follows:

According to the relative values of ((cid:1863)(cid:2869)(cid:2870) − (cid:2010)(cid:2870)), solutions to Eq. (16) can be classified into three types.

Case 1(a) ((cid:1863)(cid:2869)(cid:2870) − (cid:2010)(cid:2870) < 0): Here, we choose ((cid:1863)(cid:2869)(cid:2870) − (cid:2010)(cid:2870))(cid:2012)(cid:2870)/

= −(cid:2869) and set (cid:2018) = (2(cid:2009) − (cid:2010))(cid:2012)⁄ so that Eq. (16) becomes:

which is known as Whittaker' s Equation [20]. The general solution is:

where (cid:1839)(cid:3089),(cid:3041)((cid:1876)) and (cid:1849)(cid:3089),(cid:3041)((cid:1876)) are Whittaker functions.

Case 1(b) ((cid:1863)(cid:2869)(cid:2870) − (cid:2010)(cid:2870) = 0): We choose (2(cid:2009) − (cid:2010)) = 4(cid:2012) and set (cid:2029) = (cid:1866) − (cid:2869)(cid:2870) so that Eq. (16) becomes:

which is related to Bessel' s equation. The general solution of Eq. (19) is:

where (cid:1836)(cid:3041)() and (cid:1851)(cid:3041)() are the Bessel function of the first and second type, respectively.

Case 1(c) ((cid:1863)(cid:2869)(cid:2870) − (cid:2010)(cid:2870) > 0): We choose ((cid:1863)(cid:2869)(cid:2870) − (cid:2010)(cid:2870))(cid:2012)(cid:2870)⁄ = 1 and set (cid:2015) = −(2(cid:2009) − (cid:2010))/2(cid:2012) and (cid:2029) = (cid:1866) − (cid:2869)(cid:2870) so that Eq. (16) becomes:

which is the Coulomb wave equation. Its general solution is:

where (cid:1832)(cid:3100)((cid:2015), (cid:1876)) is the regular Coulomb wave function and (cid:1833)(cid:3100)((cid:2015), (cid:1876)) is the irregular Coulomb wave function.

Case 2: In this case, the shear modulus is a Gaussian function of r and (cid:1863)(cid:2870)((cid:1870)) is a linear function of (cid:1870)(cid:2870) as follows:

where (cid:2020)(cid:2869), , (cid:1863)(cid:2869), (cid:2870), and are constants that are determined in the same manner as in Case 1. The expression for (cid:1837)((cid:1870)) is:

So the corresponding Eq. (10) becomes:

We make a similar substitution as in Case 1 to solve this problem:

with

Similarly, (cid:2012) is a parameter that can be selected at our disposal. Eq. (25) is transformed to:

As before, according to the relative values of ((cid:2011) − (cid:2010)(cid:2870)), solutions to Eq. (27) can also be classified into three types.

Case 2(a) ((cid:2011) − (cid:2010)(cid:2870) < 0): We choose ((cid:2011) − (cid:2010)(cid:2870))(cid:2012)(cid:2870)⁄ = −1 and set (cid:2018) = ((cid:1863)(cid:2869)(cid:2870) + 2(cid:2010))/4(cid:2012) so that Eq. (27) becomes:

the general solution to which is:

Case 2(b) ((cid:2011) − (cid:2010)(cid:2870) = 0): We choose (cid:1863)(cid:2869)(cid:2870) + 2(cid:2010) = (cid:2012) and set (cid:2029) = (cid:3041)(cid:2879)(cid:2869)(cid:2870) so that Eq. (27) becomes Eq. (19), the general solution to which is:

Case 2(c) ((cid:2011) − (cid:2010)(cid:2870) > 0): We choose (cid:2011) − (cid:2010)(cid:2870) = 4 (cid:2870) and set (cid:2015) = −(cid:3038)(cid:3117)(cid:3118)(cid:2878)(cid:2870)(cid:3081)(cid:2876)(cid:3083) and (cid:2029) = (cid:3041)(cid:2879)(cid:2869)(cid:2870) so that Eq. (27) becomes Eq. (21). The general solution of Eq. (27) is thus:

4. Analytical Solutions of Single Scattering Problems

[Figure 1: see original paper] Schematic of single scattering problem with a radially gradient interphase layer

In this work, the single scattering of SH waves by a cylindrical fiber with the two interphase layers discussed above was investigated. Figure 1 shows a schematic diagram of the single scattering by a cylindrical fiber with a radially gradient interphase layer. The inner radius of the interphase layer is a and the outer radius is b. The incident wave is a plane SH wave propagating in the positive x direction with a unit magnitude.

Under the excitation of this incident wave, the total wave in the matrix, interphase layer, and the fibers can be expressed as follows:

where (cid:2013)(cid:2868) = 1 and (cid:2013)(cid:3041) = 2 ((cid:1866) 0), (cid:1853) is the nth order Hankel function of the first type; (cid:1854) and (cid:1875)(cid:3036)(cid:3041)(cid:3030)((cid:1876)) are two functions with detailed expressions dependent on the type and material properties of the interphase layer. In Case 1, (cid:1854) and (cid:1853) can be expressed as follows:

In Case 2, (cid:1854) and (cid:1853) are:

where (cid:1827)(cid:3041), (cid:1828)(cid:3041), (cid:1829)(cid:3041), and (cid:1830)(cid:3041) are the expansion coefficients to be determined by the continuity conditions of displacements and stresses across the interfaces at (cid:1870) = (cid:1853) and b:

As the expansion coefficients (cid:1827)(cid:3041), (cid:1828)(cid:3041), (cid:1829)(cid:3041), and (cid:1830)(cid:3041) are obtained, the far-field scattering magnitude [13] and the scattering cross section [8] can be calculated as follows:

For the sake of illustration, the single scattering of SH waves by a SiC fiber embedded in an aluminum matrix with a radially gradient interphase layer was solved in this work. The material properties are listed in Table 1 . Figure 2 [Figure 2: see original paper] plots the calculated (cid:2026)(cid:3046) changes with (cid:1863)(cid:3040)(cid:1853) for the interphase types (Case 1 and Case 2). The results of the approximate approach (i.e., dividing the interphase layer into multiple homogeneous sub-layers) are also plotted for comparison. Both approaches yield similar results.

[Figure 2: see original paper] Comparison of (cid:2026)(cid:3046) between current theoretical solutions and approximate method for interphase layer types (Case 1, Case 2) at several frequencies. Note: a=71 m and interphase layer thickness is h = 0.1 a

5. Conclusions

This paper reported the analytic solutions to the single scattering of SH waves by a cylindrical fiber with two specific radially gradient interphase layers, using the method proposed by P.A. Martin. In the first case, the shear modulus is an exponential function of r and (cid:1863)(cid:2870) is a linear function of 1/r; in the second case, the shear modulus is a Gaussian function of r and (cid:1863)(cid:2870) is a linear function of (cid:1870)(cid:2870). Besides SH waves, the extension of the current method to other waves has also been mentioned.

An example of the single scattering of SH waves by a SiC fiber in an aluminum matrix was calculated and the scattering cross sections were compared against the values obtained by dividing the continuously varying interphase layer into multiple homogeneous sub-layers. The two approaches yielded similar results. The main contribution of this work is that it provides several benchmark solutions for inhomogeneous problems.

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Appendix: General Approach to Make the Second Transformation

Define

with

The first and second derivatives of (cid:1870) with regard to r can then be expressed as

Substitution of the expressions of (cid:1870) into Eq. (14), we can obtain

Therefore, if Eq. (A.2) is required to have the form of the confluent hypergeometric equation, the coefficients c and d should satisfy

The term of the first derivative of (cid:1870) disappears. Then Eq. (A.2) becomes

Then d can be set to be 1 and c = −. Then Eq. (A.4) is simplified as

Note: Figure translations are in progress. See original paper for figures.

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