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Abstract

The generalized Rabi model is exactly solved by employing the unitary transformation method in the occupation number representation. The analytical expressions for the complete energy spectrum consisting of two sub-energy spectra are presented in the whole range of all the physical parameters. Each energy level possesses twofold degeneracy and is determined by the parameter in the unitary transformation, which obeys a highly nonlinear equation. The corresponding eigenfunction is a convergent infinite series in terms of these physical parameters. Due to the level crossings between the neighboring eigenstates at certain parameter values, such the degeneracies could lead to novel physical phenomena in these quantum systems with the light-matter interaction.

Full Text

Exact Solution of the Generalized Rabi Model

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The generalized Rabi model is exactly solved by employing the unitary transformation method in the occupation number representation. Analytical expressions for the complete energy spectrum, consisting of two sub-energy spectra, are presented across the entire range of physical parameters. Each energy level possesses twofold degeneracy and is determined by a parameter in the unitary transformation that obeys a highly nonlinear equation. The corresponding eigenfunction is a convergent infinite series in terms of these physical parameters. Due to level crossings between neighboring eigenstates at certain parameter values, such degeneracies could lead to novel physical phenomena in quantum systems with light-matter interaction.

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The Rabi model describes a two-level system coupled with a single bosonic mode [**?**]. This simplest interacting quantum model has found wide applications in many fields of physics, including atomic physics [**?**], quantum optics [**?**], trapped ions [**?**, **?**], quantum dots [**?**], superconducting qubits [**?**, **?**, **?**], cold atoms [**?**], and others. It is also expected to serve as the theoretical basis for quantum information and quantum technology [**?**].

A generalized Rabi model typically has the Hamiltonian

$$H = \omega a^{+}a + g(a^{+} + a)\sigma_{x} + \lambda\sigma_{z} + \epsilon\sigma_{x},$$

where σ_x and σ_z are Pauli matrices for the two-level system with level splitting 2λ, a^+ and a are creation and annihilation operators for the single bosonic mode with frequency ω, respectively, the light-matter interaction is controlled by the coupling parameter g, and the last term $\epsilon\sigma_x$ is the driving term that breaks the Z_2 symmetry and leads to tunneling between the two levels. We note that the competition between g and ω produces different experimental regimes. When g/ω is small, applying the rotating-wave approximation to the Rabi model (1) with $\epsilon = 0$ leads to the Jaynes-Cummings model [**?**], which is relevant to most experimental regimes. Because the Jaynes-Cummings model is integrable, its analytical solution is easily derived. With increasing g/ω, the ultrastrong coupling regime ($g/\omega \gtrsim 0.1$) [**?**] or the deep strong coupling regime ($g/\omega \gtrsim 1.0$) [**?**] is reached, where the Jaynes-Cummings model becomes invalid and cannot be used to investigate the interaction between light and matter. Recently, these regimes have attracted rapidly growing interest due to their fundamental characteristics and potential applications in quantum devices [**?**].

Although the Hamiltonian (1) has a simple form, obtaining its analytical solution has not been possible until now, despite being considerably important for accurately exploring light-matter interaction from weak to extremely strong coupling. In Ref. [**?**], Braak presented an analytical solution of the Rabi model using the representation of bosonic operators in the Bargmann space of analytical functions. The energy spectrum consists of two parts: the regular and exceptional spectra. However, such a spectrum structure is considerably strange. It has been proved that Braak' s analytical solution of the Rabi model is completely wrong due to a derivation error in solving the time-independent Schrödinger equation [**?**].

In this work, we exactly diagonalize the Hamiltonian (1) across the entire range of physical parameters using the unitary transformation technique in the occupation number representation. This direct and powerful approach has been successfully used to solve the complex two-dimensional electron gas in the presence of both Rashba and Dresselhaus spin-orbit interactions under a perpendicular magnetic field [**?**]. The eigenvalues are fixed by parameters in the unitary transformations that satisfy highly nonlinear equations. The eigenfunctions are

convergent infinite series in terms of the physical parameters. The exact Landau levels have been widely accepted [**?**] (also see the citations of Ref. [**?**]). When the Rashba or Dresselhaus spin-orbit coupling vanishes, this two-dimensional electron system has the same bosonic Hamiltonian as the Jaynes-Cummings model except for a constant [**?**, **?**].

The two-component eigenstate of the Hamiltonian (1) for the nth energy level with quantum number s typically has the general form

$$|n,s\rangle = \frac{1}{\sqrt{1+\Delta_{ns}^2}} \begin{pmatrix} \sum_{m=0}^{\infty} \alpha_{ns}^m \phi_m \\ \Delta_{ns} \sum_{m=0}^{\infty} \beta_{ns}^m \phi_m \end{pmatrix},$$

where the 2×2 matrix is unitary, $s = \pm 1$, Δ_{ns} is a real parameter to be determined below by requiring α_{ns}^m and β_{ns}^m to be nonzero, ϕ_m is the eigenstate of the mth energy level in the occupation number representation (i.e., $a^+\phi_m = \sqrt{m+1}\phi_{m+1}$, $a\phi_m = \sqrt{m}\phi_{m-1}$, and $\langle \phi_{m'}|\phi_m\rangle = \delta_{mm'}$), and $\alpha_{ns}^m = 0$ when $m \to +\infty$.

We solve the eigen-equation $H|n,s\rangle = E_{ns}|n,s\rangle$ and obtain a doubly degenerate energy spectrum consisting of two sub-energy spectra, which is very different from that presented in Ref. [**?**].

Sub-Energy Spectrum I

To obtain the analytical solution of the Hamiltonian (1) in the whole parameter space, we first choose

$$[\omega(n+1) + \frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda - \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon - E_{ns}]\alpha_{ns}^n + g\sqrt{n+1}\beta_{ns}^{n+1} = 0,$$

$$[\omega n - \frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda + \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon - E_{ns}]\beta_{ns}^n + g\sqrt{n+1}\alpha_{ns}^{n+1} = 0$$

in the eigen-equation. Then the eigenfunction associated with the eigenvalue E_{ns} is uniquely fixed by requiring

$$[2\lambda\Delta_{ns} + \epsilon(1-\Delta_{ns}^2)]\beta_{ns}^n - 2g\Delta_{ns}\sqrt{n+1}\alpha_{ns}^{n+1} = 0, \tag{4}$$

$$[2\lambda\Delta_{ns} + \epsilon(1-\Delta_{ns}^2)]\alpha_{ns}^{n+1} + 2g\Delta_{ns}\sqrt{n+1}\beta_{ns}^n = 0. \tag{5}$$

We solve the homogeneous linear equations (3) about α_{ns}^{n+1} and β_{ns}^n by vanishing the coefficient determinant. The eigenvalue for the nth eigenstate with s then has the analytical expression

$$E_{ns} = \left(n + \frac{1}{2}\right)\omega + s\Xi_{ns}, \tag{6}$$

where

$$\Xi_{ns} = \sqrt{\left(\frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda - \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon\right)^2 + (n+1)g^2\left(\frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\right)^2}.$$

Note that the quasiparticle energy E_{ns} must be larger than zero. Combining Eqs. (3)-(5), the parameter Δ_{ns} is determined by the constraint

$$\epsilon(1+\Delta_{ns}^2) - 2\Delta_{ns}(E_{ns} - \omega n) = 0, \tag{7}$$

$$\epsilon(1+\Delta_{ns}^2) + 2\Delta_{ns}[E_{ns} - \omega(n+1)] = 0. \tag{8}$$

Surprisingly, Eq. (7) with $s = -1(1)$ coincides with Eq. (8) with $s = 1(-1)$. Therefore, we have

$$\epsilon(1+\Delta_{ns}^2) + \Delta_{ns}(2\sigma\Xi_{ns} - \omega) = 0, \tag{9}$$

where $\sigma = \pm 1$. Obviously, when $\epsilon = 0$, then $\Delta_{ns} = 0$ from Eq. (9). Consequently, the eigenvalue (6) has a simple form in the absence of the driving term.

[Figure 1: see original paper] shows the low-lying energy levels of the sub-energy spectrum I in units of ω as a function of the coupling parameter g at different λ under $\epsilon = 0$. The solid lines denote $n = 0, 1, \cdots, 5$ and $s = 1$, while the dashed lines represent $n = 1, 2, \cdots, 5$ and $s = -1$. We observe level crossings between neighboring eigenstates. With increasing λ, the energy levels with $s = 1(-1)$ become higher (lower), and these crossing points move toward the origin.

When $\epsilon \neq 0$, Δ_{ns} in Eq. (9) with $\sigma = 1$ has an ω-dependent solution. The corresponding eigenvalues E_{ns} ($n = 0, 1, 2, \cdots, \infty$, $s = \pm 1$) form the sub-energy spectrum I. In [Figure 2: see original paper], we show the low-lying energy levels of the sub-energy spectrum I as a function of g at different λ under $\epsilon = 0.4$, along with the corresponding parameter Δ_{ns}. From [Figure 1: see original paper] and [Figure 2: see original paper], we find that if λ or ϵ increases, the energy levels with $s = 1(-1)$ also move up (down), and the crossing points shift toward the left side. When $g = 0$, the sub-energy spectrum I (6) recovers the exact eigenvalues for the interactionless case:

$$E_{ns} = \omega n + s\sqrt{\lambda^2 + \epsilon^2} \quad \text{with} \quad \Delta_{ns} = \frac{\lambda - \sqrt{\lambda^2 + \epsilon^2}}{\epsilon}.$$

Obviously, if $\epsilon \to 0$, then $\Delta_{ns} \to 0$. We note that another sub-energy spectrum corresponding to the ω-dependent solution Δ_{ns} in Eq. (9) with $\sigma = -1$ is nothing but the sub-energy spectrum II (see below).

For the eigenstate associated with eigenvalue (6), we have

$$\beta_{ns}^{n} = \frac{(1+\Delta_{ns}^2)(E_{ns} - n\omega) + (1-\Delta_{ns}^2)\lambda - 2\Delta_{ns}\epsilon}{2g\Delta_{ns}\sqrt{n+1}},$$

where β_{ns}^{n} is an arbitrary constant that can be set to 1. The coefficients α_{ns}^{i} and β_{ns}^{i} are uniquely determined by the recursion relations

$$\alpha_{ns}^{i} = -\frac{N_{ns}^{i+1}}{M_{ns}} \quad \text{for} \quad i = 0, 1, 2, \cdots, n,$$

and

$$\beta_{ns}^{i+1} = -\frac{N_{ns}^{i}}{M_{ns}} \quad \text{for} \quad i = n+1, n+2, \cdots, +\infty,$$

where we have defined

$$M_{ns} = \left[\frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\sigma_x - \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\sigma_z\right],$$

$$N_{ns}^{i} = \left[(\omega i - E_{ns})I + \left(\frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\lambda + \frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\epsilon\right)\sigma_x + \left(\frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda - \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon\right)\sigma_z\right],$$

with I being the 2×2 unit matrix.

Sub-Energy Spectrum II

Now we use another choice:

$$[\omega n + \frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda + \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon - E_{ns}]\alpha_{ns}^{n} + g\sqrt{n+1}\beta_{ns}^{n+1} = 0,$$

$$[\omega(n+1) - \frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda - \frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon - E_{ns}]\beta_{ns}^{n} + g\sqrt{n+1}\alpha_{ns}^{n+1} = 0,$$

in the eigen-equation. The corresponding eigenfunction is uniquely determined by letting

$$[2\lambda\Delta_{ns}+\epsilon(1-\Delta_{ns}^2)]\alpha_{ns}^n+2g\Delta_{ns}\sqrt{n+1}\beta_{ns}^{n+1}=0, \tag{15}$$

$$[2\lambda\Delta_{ns}+\epsilon(1-\Delta_{ns}^2)]\beta_{ns}^{n+1}-2g\Delta_{ns}\sqrt{n+1}\alpha_{ns}^n=0. \tag{16}$$

Solving Eqs. (14)-(16), we obtain the eigenvalue for the nth eigenstate with s:

$$E_{ns}=\left(n+\frac{1}{2}\right)\omega+s\Theta_{ns}, \tag{17}$$

where

$$\Theta_{ns}=\sqrt{\left(\frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\lambda+\frac{2\Delta_{ns}}{1+\Delta_{ns}^2}\epsilon\right)^2+(n+1)g^2\left(\frac{1-\Delta_{ns}^2}{1+\Delta_{ns}^2}\right)^2}.$$

Here Δ_{ns} satisfies the nonlinear equation

$$\epsilon(1+\Delta_{ns}^2)+2\Delta_{ns}(E_{ns}-\omega n)=0, \tag{18}$$

$$\epsilon(1+\Delta_{ns}^2)-2\Delta_{ns}[E_{ns}-\omega(n+1)]=0, \tag{19}$$

which is derived from Eqs. (14)-(16). Similar to Eqs. (7) and (8), Eq. (18) with $s=-1(1)$ is also consistent with Eq. (19) with $s=1(-1)$. Therefore, we obtain

$$\epsilon(1+\Delta_{ns}^2)+\Delta_{ns}(2\tau\Theta_{ns}+\omega)=0, \tag{20}$$

where $\tau=\pm1$. If $\epsilon=0$, then $\Delta_{ns}=0$ in Eq. (20). Thus, the eigenvalue (17) also has an explicit expression in this case. The low-lying energy levels as a function of g at different λ under $\epsilon=0$ are depicted in [Figure 3: see original paper]. With increasing λ, the energy levels with $s=-1(1)$ become higher (lower) while the crossing points move away from the origin, opposite to the behavior in [Figure 1: see original paper].

If $\epsilon\neq0$, Δ_{ns} in Eq. (20) with $\tau=1$ also has an ω-dependent solution. The corresponding eigenvalues constitute the sub-energy spectrum II. [Figure 4: see original paper] exhibits the low-lying energy levels of the sub-energy spectrum II as a function of g at different λ under $\epsilon=0.4$, along with the corresponding parameter Δ_{ns}. When λ increases, the energy levels with $s=1(-1)$ also move up (down), and the crossing points shift toward the origin, resembling the level shifts in the sub-energy spectrum I (see [Figure 2: see original paper]). From [Figure 3: see original paper] and [Figure 4: see original paper], we find

that when $\lambda = 0.3\omega$, the energy levels with $s = -1(1)$ move up (down) with increasing ϵ. Conversely, the energy levels with $s = 1(-1)$ move up (down) at $\lambda = 0.5\omega$. Obviously, the level shifts in the sub-energy spectrum II are more complex than those in the sub-energy spectrum I. We note that the sub-energy spectrum associated with Δ_{ns} in Eq. (20) with $\tau = -1$ is consistent with the sub-energy spectrum I. Therefore, both the sub-energy spectrum I and II are doubly degenerate, meaning Braak' s mathematical solution also loses half of the eigenvalues.

For the nth eigenstate with s in the sub-energy spectrum II, we have

$$\alpha_{ns}^{n} = \frac{(1+\Delta_{ns}^{2})(E_{ns}-n\omega)-(1-\Delta_{ns}^{2})\lambda+2\Delta_{ns}\epsilon}{2g\Delta_{ns}\sqrt{n+1}},$$

where α_{ns}^{n} is an arbitrary constant set to 1. The coefficients α_{ns}^{i} and β_{ns}^{i} obey the same recursion relations (11) and (12) as in the sub-energy spectrum I.

Summary and Outlook

In summary, we have exactly solved the generalized Rabi model (1) within the framework of quantum mechanics. The complete spectrum comprises two twofold degenerate sub-energy spectra, I and II. Very different from the energy spectrum in Ref. [**?**], level crossings occur even when ϵ is not a multiple of $\omega/2$. Such physical solutions can help us deeply understand light-matter interaction, especially in the strong coupling regime. It is expected that level crossings at certain physical parameters could produce novel physical phenomena in these interacting systems, just as in two-dimensional electron gases with spin-orbit interaction under a perpendicular magnetic field [**?**]. Finally, we would like to mention that the unitary transformation method presented here can be naturally applied to the general Rabi model, which is a further extension of the generalized Rabi model obtained by adding $H' = (a^{+}+a)(g_y\sigma_y + g_z\sigma_z) + \epsilon_y\sigma_y$ to the Hamiltonian (1), where g_y, g_z, and ϵ_y are physical parameters.

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