Holography and Entanglement in Flat Spacetime (Postprint)

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Abstract

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Full Text

Holography and Entanglement in Flat Spacetime

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Abstract. We propose a holographic correspondence for flat spacetime based on the behavior of entanglement entropy and correlation functions. The holographic dual theory turns out to be highly nonlocal. We argue that after most of the space is traced out, the reduced density matrix gives maximal entropy and the correlation functions become trivial. We present a toy model for this holographic dual using a nonlocal scalar field theory that reproduces the same property of entanglement entropy. Our conjecture is consistent with the entropy of Schwarzschild black holes in asymptotically flat spacetimes.

Introduction

One of the most powerful tools to study quantum gravity is the holographic duality conjecture: quantum gravity in spacetime is equivalent to a quantum field theory living on the boundary M [1]. In particular, quantum gravity in anti-de Sitter space has been well developed via the AdS/CFT correspondence [2] from string theory. However, to understand our universe, we need to study quantum gravity in other spacetimes such as flat space, de Sitter space, and the big bang spacetime. The purpose of this Letter is to investigate holography in flat spacetime and to present a consistent outline of its basic properties and mechanism.

We will focus on Euclidean flat spacetime R^{d+1} since the Euclidean formulation is often simpler and better defined than the Lorentzian version, as in AdS/CFT [3]. In polar coordinates, the metric of Euclidean spacetime R^{d+1} is ds^2 = d ^2 + ^2 ds_{S^{d}}2. The holographic principle dictates that the boundary dual theory of gravity in R^{d+1} lives on the unit-radius sphere S^d at = _∞, where _∞ is the bulk cutoff radius and is related to the UV cutoff in the boundary field theory (as in AdS/CFT). We take the limit _∞ → ∞.

The assumptions we adopt in this Letter are the following: (1) the dual field theory allows a path-integral formulation even if it is nonlocal; (2) the bulk-to-boundary correspondence holds, i.e., the partition function of gravity in R^{d+1} equals that of the holographic dual theory on S^d [3]. In Lorentzian holography, S^d is replaced by d-dimensional de Sitter space (see [4, 5] for earlier studies). See [6–8] for other approaches to holography in flat space.

Holographic Entanglement Entropy

When a quantum system is divided into two subsystems A and B, the von Neumann entropy S_A = -Tr[_A log _A] (where _A is the reduced density matrix after tracing out B) is called the entanglement entropy of subsystem A. The scaling behavior and certain universal coefficients of the entanglement entropy encode important information about the degrees of freedom and nonlocal correlations of the system. Furthermore, the entanglement entropy is a general-purpose quantity since it can be defined in any quantum many-body system that allows a path-integral formalism—even in nonlocal field theories, as will be shown later. Thus the entanglement entropy is particularly useful when we know little else about the holographic dual of a given gravity theory, as in our case.

On the gravity side, there is a general prescription to compute the entanglement entropy holographically: when the d-dimensional boundary system is divided into two parts A and B, the holographic entanglement entropy of subsystem A is given by the formula [9]

S_{hol} = area(γ*A) / (4 G*{d+1})

where area(γ_A) is the area of the minimal surface γ*A that lies inside the (d+1)-dimensional bulk and borders on the boundary A of subsystem A; G*{d+1} is the (d+1)-dimensional Newton’ s constant.

Now we apply this to compute the holographic entanglement entropy of a Euclidean field theory living on the boundary of R^{d+1}. The metric of the boundary sphere S^d is $ds^2 = d\tau^2 + \cos^2 \tau \; d\Omega_d^2$, where τ [$-\pi/2$, $\pi/2$] is regarded as Euclidean time, and the spatial slice of constant τ is S^{d-1}, whose metric can be written as $ds^2 = d\ ^2 + \sin^2 \ \ d\Omega_{d-1}^2$.

We divide the spatial slice S^{d-1} at $\tau = \tau_0$ into two spherical caps A and B using a subsphere S^{d-2} given by $= \ _0$. The radius of this S^{d-2} is $\sin \alpha$, where α is the geodesic distance in S^d between antipodal points of S^{d-2}, with $\cos \alpha = \cos \tau_0 \sin \ _0$ (see [Figure 1: see original paper]). The holographic entanglement entropy is

$$S_{hol} = (\Gamma((d+1)/2) \; / \; (4 \, G_{d+1} \sqrt{\pi} \, \Gamma(d/2))) \cdot (\ _\infty^d \sin^d \alpha)$$

First we notice that for a small subsystem A ($\alpha \to 0$), S_{hol} approaches A' s volume instead of its area. Moreover, S_A with generic α is extensive since it is proportional to the spatial volume of the full boundary system (see also [10]). These two facts are in sharp contrast with the behavior both in local field theories [11] and in AdS/CFT. In a local field theory at its ground state, the leading divergence of the entanglement entropy is always proportional to the surface area of the subsystem (the "area law") [11]. The entropy becomes extensive only when the system is in highly excited states with energy around the UV cutoff [12, 13].

Toy Model: Nonlocal Scalar Field Theory

We have used a free scalar theory to show that a nonlocal theory is needed to realize the volume law for entanglement entropies. The full holographic dual of flat space is likely to contain more fields and to be strongly-interacting. As in AdS/CFT and in standard entanglement entropy computations, adding more fields and turning on local interactions do not alter the scaling behavior of the entanglement entropy; we expect that appropriate nonlocal interactions preserve the volume law as well.

Let us consider a generic (not necessarily local) free scalar field theory on S^d defined by the action

$$S_{boundary} = \ \ d\Omega_d \, [\phi \, f(-\Delta) \, \phi]$$

where Δ is the Laplacian on S^d and f(x) is a smooth function (see [15] for an analogous computation in R^d). To see the extensive behavior of the entanglement entropy, it suffices to consider the simplest configuration with $\alpha = \pi$. In this case, S_A can be expressed as follows (similar to the geometric entropy in [16]):

$$S_A = - \textit{N log } Z_N / \{N=1\}$$

where Z_N is the partition function of the action on the orbifold S^d/Z_N; the Z_N action is defined by a $2\pi/N$ rotation of S^d.

The partition function can be evaluated via Schwinger representation:

$$\log Z_N = -1/2 \; _{}^{\infty} (ds/s) \, \mathrm{Tr}(N) \, e^{-s f(-\Delta)}$$

where is related to the UV cutoff in the field theory. Spherical harmonics on S^d are labeled by angular momenta $(l, m_1, \cdots, m_{d-1})$, which range as $l \geq 0$ and $|m_i| \leq l$. The eigenvalues of the Laplacian Δ are $l(l + d - 1)$. The Z_N orbifolding acts by multiplying a phase factor $e^{2\pi i\, k/N}$ $(k = 1, \cdots, N)$ to each spherical harmonic. The relevant trace $\mathrm{Tr}(N) \, e^{-s f(-\Delta)}$ is then:

$$\mathrm{Tr}(N) \, e^{-s f(-\Delta)} = \Sigma_{k=1}^{N} \Sigma_{l=0}^{\infty} g(l, d, k/N) \, e^{-s f(l(l+d-1))}$$

where $g(l, d, k/N)$ incorporates the sum over all magnetic angular momenta m_i and is computed to be:

$$g(l, d, k/N) = (2l + d - 1) \, (l + d - 2)! \, / \, (l! \, (d - 1)!) \cdot (\sin[\pi k(l + (d-1)/2)/N] \, / \, \sin(\pi k/N))$$

For $d = 2$, the binomial coefficient simplifies and $g(l, 2, k/N) = \sin[\pi k(2l+1)/N] \, / \, \sin(\pi k/N)$. Lower dimensional spheres have more compact results, while higher dimensional spheres $(d \geq 4)$ have more complicated expressions that need to be treated separately.

Summing over all twisted sectors using $\Sigma_{k=1}^{N-1} \sin^{-2}(\pi k/N) = (N^2 - 1)/6$ and then applying $_N$, we obtain the leading divergence of the entanglement entropy:

$$S_A \quad (V_{S^d} \, / \, (2(4\pi)^{d/2} \, \Gamma(d/2))) \quad ^{\infty} \, ds \; s^{-d/2-1} \, f' \, (l(l+d-1))/_{l \to \infty} + \cdots$$

Now we impose the UV cutoff. For S^d with radius L and lattice spacing a, the azimuthal angular momentum l has an upper bound given by $l_{max} = L/a$. This translates into a lower bound on the integration parameter s: $s \geq 1/f(l_{max}^2)$.

First, let us look at actions with $f(x) = x^p$; in particular, $p = 1$ gives the standard massless scalar. The leading divergence of the entanglement entropy is:

$$S_A \quad (V_{S^d} \, / \, (2(4\pi)^{d/2} \, \Gamma(d/2) \, p)) \cdot (L/a)^{d-2p} \, / \, (d - 2p)$$

Although this result is obtained for $d \geq 4$, an exact computation for $d = 2, 3$ shows that it actually holds for all $d \geq 2$. In particular for $d = 2$, this gives $S_A = (1/p) \log(L/a)$ (after an infinite constant term is dropped). Therefore all theories with $f(x) = x^p$ are local and obey the area law.

Now we make the theory nonlocal by choosing $f(x) = x^{1/2 + q}$. For all S^d with $d \geq 2$, the leading divergence becomes:

$$S_A \quad (V_{S^d} \, / \, (4\sqrt{\pi} \, \Gamma(d/2) \, (1 + 2q))) \cdot (L/a)^{d-1-2q}$$

Therefore S_A obeys the volume law when $q = 1/2$, in any dimension d. To summarize, we find that a nonlocal scalar field theory defined by the action with

$f(x) = x$ has entanglement entropies that exhibit the volume law.

The bulk Bekenstein bound requires that the maximal boundary statistical entropy is bounded by the volume when the system is in highly excited states with energy around the UV cutoff [12, 13]. However, the holographic dual of gravity in flat space should not be restricted to any particular type of states in the boundary theory since this "volume law" applies to the holographic entanglement entropy of any asymptotically flat space. Taking all of these into account, we conjecture that the holographic dual is described by a certain nonlocal field theory. Below we will construct one such example based on scalar field theory.

Also note that in the $\alpha \to 0$ limit, S_A saturates the holographic bound [14] (here given by the volume of A in S^d). Therefore although our total system is in a pure state as evidenced by $S_A = S_B$, an infinitesimal subsystem A has a density matrix $\_A$ with maximal entropy (= log dim H_A, where H_A is the Hilbert space of A). Namely, an infinitesimal subsystem A is maximally entangled with its complement B.

Holographic Correlation Functions

Another important quantity in establishing holography is the correlation function. Now we extend the bulk-to-boundary procedure [3] to flat space and compute its holographic correlation functions.

Consider a scalar field theory in the bulk:

$$S_{bulk} = (1/(32\pi G_{d+1}))\quad d^{d+1}x \sqrt{g}\ (\_\ \phi\ \hat{}\ \ \phi + M^2 \phi^2)$$

Under the Dirichlet boundary/bulk correspondence, we have:

$$D\Phi \exp(i\, S_{boundary}[\Phi]) = Z_{R^{d+1}}[\phi(\_\infty, \Omega) = \Phi(\Omega)]$$

where the left-hand side is the generating functional of correlation functions on the boundary S^d and the right-hand side is the bulk partition function under the Dirichlet boundary condition; Φ is the boundary scalar field, which sources the operator $\hat{O}$ in S^d.

In the classical limit, $Z_{R^{d+1}}[\Phi]\quad \exp(i\, S_{bulk}[\phi_{cl}, \Phi])$. Since the boundary theory decouples from gravity in the limit $G_{d+1} \to 0$, Newton' s constant on the boundary is $G_d = G_{d+1}\ (\_\infty)^{d-1}$.

For the massless scalar, $\phi(, \Omega)$ can be solved using the bulk Dirichlet Green' s function. The on-shell action gives the boundary two-point function:

$$\hat{O}(\Omega_1)\ \hat{O}(\Omega_2)\quad (1/(32\pi G_{d+1})) \cdot (A_d / (\cos\ \ - 1)^{d})$$

where $\cos\ \ = \Omega_1 \cdot \Omega_2$ and A_d is the area of the unit sphere S^d. This agrees with the analysis in [20], though our interpretations are slightly different. Since this expression contains only a divergent term, the physical two-point function vanishes after nonlocal boundary counterterms are added to cancel the divergence. Nonlocal counterterms were also used in the holography of NS5-branes [4, 21].

For the massive scalar, we decompose the boundary field as:

$$\Phi(\Omega) = \Sigma_{l, \bar{m}} c_{l, \bar{m}} Y_{l, \bar{m}}(\Omega)$$

where $Y_{l, \bar{m}}(\Omega)$ are orthonormal spherical harmonics on S^d. Then the same bulk-to-boundary procedure produces the boundary two-point function:

Ô(Ω_1) Ô(Ω_2)   (1/(32πG_{d+1})) Σ_{l, m̄} (log(^{d/2} I{l+(d-1)/2}(M)))|{ →∞} Y{l, m̄}(Ω_1) Y_{l, m̄}(Ω_2)

where I_ (z) is the modified Bessel function of the first kind. Since log(^{d/2} I{l+(d-1)/2}(M)) as a polynomial of has a nonzero constant term, which is a degree [d/2] polynomial of l(l + d - 1) (eigenvalue of Δ), and using the identity Σ_{l, m̄} Y{l, m̄}(Ω_1) Y{l, m̄}(Ω_2) = δ(Ω_1 - Ω_2), we see that in the limit _∞ → ∞, after counterterms are added to cancel the divergence, the two-point functions consist of δ-functions and their derivatives by Laplacian. Therefore, the holographic correlation functions for a massive scalar are essentially zero.

Next one could explicitly compute higher-point functions following the bulk-to-boundary principle. However, if we assume the dilaton-type massless scalar Lagrangian of the form L = (ϕ)^2 P(ϕ) where P(ϕ) is a polynomial, the correlators always scale as ∞^d. Therefore they can all be eliminated by adding boundary counterterms.

In summary, we argue that all n-point correlation functions vanish after counterterms are added to cancel the divergences. This seems surprising until one recalls our previous observation from the holographic entanglement entropy: A is maximally entangled with B when the size of A approaches zero. Define an infinitesimal subsystem A as the disjoint union of the n points in the correlation function: A = {i=1}^n x_i. Our previous result implies that in this case the entanglement entropy S_A is maximal, therefore the density matrix _A factorizes into a direct product _A = {i=1}^n _{x_i}, where _{x_i} gives the maximal entropy for the subsystem at point i, as in a system at infinitely high temperature. Therefore all correlation functions vanish:

Ô(x_1) ⋯Ô(x_n) = Tr[_A Ô(x_1) ⋯Ô(x_n)] = 0

We emphasize that this does not mean the boundary theory is empty: it stems from the fact that the boundary theory is nonlocal and highly entangled. Based on this we propose that the bulk physics in flat space should be reproduced by the boundary entanglement entropies (with all possible subsystems traced out).

The correlators of the free scalar toy model are not exactly zero, but the usual divergence (when two operators coincide) already disappears. This leads us to expect that choosing appropriate interactions can further reduce the correlators to zero. Indeed, such a theory already exists in discretized form: consider a spin model with the randomized antiferromagnetic Heisenberg interaction H = J Σ_{ i,j } $\hat{\sigma}$_i · $\hat{\sigma}$_j, where J > 0 and the sum is over randomly chosen spin pairs. Since the distance and orientation between two spins inside a pair is randomly distributed, all correlators are zero; and since generically the two spins in a given pair are separated far from each other, the entanglement entropy obeys the volume law. The continuum limit of this type of spin model would provide field theory candidates for the holographic dual of flat space.

Discussion

Now we draw an analogy between the boundary entanglement entropy and the bulk Schwarzschild entropy in the spirit of the connection between Unruh effect and Hawking radiation. In the Lorentzian version of the metric given by ds^2 = d ^2 + $^{2(\text{-dt}}$2 + cosh^2 t dΩ_{d-1}^2), a static observer at = _0 detects a thermal state at the Unruh temperature T_U = 1/(2π _0). The entanglement entropy S_A for maximal size A ($\alpha = \pi$) can be rewritten as:

S_A = (2^{d-1} π^{(d+1)/2} Γ((d+1)/2) / (d+1)) · (_0^d / G_{d+1}) T_U^{d-1}

Since S_A measures the amount of information hidden in subsystem B, which is inaccessible to observers in A, it is analogous to the entropy of a Schwarzschild black hole with temperature T_{BH} = T_U. Indeed, the (d+1)-dimensional Schwarzschild black hole has an entropy S_{BH} = (2^{d-1} π^{(d+1)/2} Γ((d+1)/2) / (d+1)) · (_0^d / G_{d+1}) T_{BH}^{d-1}, which agrees with S_A up to a numerical constant. Thus our holographic interpretation is consistent with black hole entropies.

This also suggests a string theory interpretation of our holography. In AdS/CFT [2], the holographic dual theory comes from D-branes that originally sit at the horizon r = 0. In our flat spacetime, in the limit _0 → 0, the Unruh temperature T_U becomes infinitely large and the corresponding observer detects pair creations of many D-branes. We speculate that their open string theory is the nonlocal field theory conjectured to be the holographic dual of flat spacetime.

Finally, let us reexamine the connection between UV cutoff in the field theory and the cutoff radius of the bulk. Matching the entanglement entropy obtained from the holographic computation (with $\alpha = \pi$) and the field theory one (with q = 1/2), we see that if we switch to dimensionless coordinates defined by ˜ = /R, where R is a length unit, and accordingly consider the boundary theory on a unit sphere S^d with dimensionless lattice spacing ã = a/R, then S_A, interpreted as the entanglement entropy for the holographic dual on the unit sphere S^d, scales as:

S_A n / ã^{d-1}

where the dimensionless number n = R^{d-1}/G_{d+1} counts the number of fields in the holographic dual. Since the bulk metric ds^2 = R$^{2(\text{d}^{\sim}}$2 + ˜^2 dΩ_d^2) is invariant under the rescaling (R, ˜) → (Rλ, ˜/λ) for arbitrary λ, there exists a corresponding symmetry in the holographic dual theory: (n, ã) → (n/λ^{d-1}, λã). Note that the total number of degrees of freedom in the boundary field theory is proportional to n/ã^{d-1} and therefore remains invariant. This symmetry suggests that the theory is highly nonlocal and entangled and will be useful when we go on to identify the precise holographic dual. We leave these questions for future study.

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