Some aspects of holographic W-gravity (postprint)

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Full Text

Preamble

Some Aspects of Holographic W-Gravity

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Abstract

We use the Chern-Simons formulation of higher spin theories in three dimensions to study aspects of holographic W-gravity. Concepts which were useful in studies of pure bulk gravity theories, such as the Fefferman-Graham gauge and the residual gauge transformations which induce Weyl transformations in the boundary theory and their higher spin generalizations, are reformulated in

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Introduction

Holography is a well-established powerful tool for detailed studies of conformal field theories. In general dimension d the CFT is dual to a gravity theory in the $(d+1)$-dimensional bulk, possibly coupled to other bulk fields, whose boundary values are sources of certain operators in the CFT. Local symmetries in the bulk are in one-to-one correspondence with global symmetries on the boundary, where they can be gauged by coupling the theory to sources of conserved currents.

The coupling of the CFT to an external metric leads to a diffeomorphism invariant theory, which, in addition, possesses classical Weyl symmetry—i.e., invariance under rescaling of the metric and possibly of the fields of the CFT. In two dimensions, to which we restrict the following discussion, the three local symmetry parameters are sufficient to gauge away the external metric. In the quantum theory, the symmetries of the classical theory cannot all be maintained simultaneously, leading to an anomaly. It manifests itself in anomalous Ward identities or in a non-invariance of the effective action, a functional of the external metric which is obtained by integrating out the quantum fields and generates correlation functions of the energy-momentum tensor.

Which of the symmetries one wants to maintain dictates the choice of the counterterms. Opting for diffeomorphism invariance (or equivalently conservation of the energy-momentum tensor) leads to the non-local Polyakov action [**?**], whose only dependence on the specific CFT is through an overall factor proportional to the central charge, which parametrizes the anomaly. In this context, the anomaly manifests itself in the non-invariance of the Polyakov action under Weyl rescaling of the metric, causing a non-vanishing vacuum expectation value of the trace of the energy-momentum tensor in an external metric background, or a non-vanishing trace of the two-point function of the energy-momentum tensor in flat space. The anomaly is also well known as the quantum mechanically induced central extension of the infinite-dimensional conformal algebra (the symmetry algebra of classical CFTs) to the Virasoro algebra.

Besides conformal symmetry, two-dimensional CFTs can also have enhanced symmetries, the most prominent ones being Kac-Moody symmetries with spin-one currents and supersymmetries with fermionic symmetry currents with spin 3/2. In this paper we are interested in CFTs with conserved higher-spin currents whose symmetry algebras are W-symmetries, which have the Virasoro algebra as a sub-algebra. The simplest known and earliest example is Zamolodchikov's W_3 algebra [**?**]. In the same way as a CFT can be coupled to an external metric, which sources the energy-momentum tensor of the CFT, W-symmetries can be coupled to higher-spin gauge fields, which source conserved higher-spin currents. This leads to the notion of W-gravity. At the classical level, the symmetries are higher-spin generalizations of diffeomorphism, parametrized by a traceless symmetric rank s tensor with two components, and generalized Weyl transformations, parametrized by a symmetric rank $s-2$ tensor with $s-1$ components. In the classical theory, these symmetries are sufficient to gauge away the $s+1$ components of the spin-s sources; but after quantization this is no longer possible. Choosing to preserve diffeomorphism invariance and higher-spin gauge symmetries results in anomalies in generalized Weyl symmetries. The anomalous symmetry transformations, called W_s-Weyl transformations, are parametrized by one scalar field for each spin s. The corresponding anomalies, which we call W_s-anomalies, manifest themselves as trace anomalies in the two-point functions of higher-spin currents or as the non-invariance of the effective action under W_s-Weyl transformations. Two-dimensional theories with W-algebras as symmetry algebra were intensively studied about 25 years ago, also in the context of string theory, but it is fair to say that the implications of the higher-spin symmetries are much less understood than those of the conformal symmetry. A good review of the early literature is [**?**].

Since then the AdS/CFT correspondence has equipped us with a new tool to study conformal field theories. To study two-dimensional conformal field theories with higher-spin W-symmetries and their couplings to higher-spin sources, we need a three-dimensional bulk theory which has, in addition to diffeomorphism, higher-spin gauge symmetries. The source for the boundary spin-s conserved current is the boundary value of the bulk gauge field of the same spin.

The AdS/CFT correspondence with higher-spin symmetry has recently been studied, in particular for the boundary dimensions $d = 2, 3$. For $d = 2$, which is the dimension we are interested in this paper, the bulk gravity theory has an alternative description as Chern-Simons theory. For 3D pure gravity, whose boundary metric sources components of the CFT energy-momentum tensor, its alternative description as $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory has been known for a long time [**?**, **?**]. More recently this was extended to include higher-spin bulk fields [**?**, **?**], where a formulation as $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ Chern-Simons theory was proposed. Corresponding to the presence of the higher-spin bulk fields the boundary theory is a CFT with higher-spin W_N-symmetry.

A bulk spin-s gauge field is realized by a symmetric space-time tensor of rank s, which we will refer to as a metric-like field. An action principle for the interacting higher-spin theory in terms of metric-like fields is not known in general. In three dimensions, since we have an alternative description in terms of Chern-Simons theory, we have, in principle, all the information at our disposal. Given a pair of flat connections, i.e., a solution of the equations of motion of the CS theory, we can construct the metric-like fields. Likewise, the higher-spin symmetries are encoded in the $sl(N)$ gauge symmetries, i.e., given a gauge symmetry we can construct the parameters of the higher-spin symmetries, which we refer to as generalized diffeomorphisms.

However, for $N > 3$, it is unclear how to translate from connections to metric-like fields at the level of the action and the equations of motion. Attempts to construct them order by order in the higher-spin fields were made, e.g., in [?, ?, ?]. One difficulty lies in determining the transformations of the metric-like fields under the generalized diffeomorphism, which at present can only be done order by order in the higher-spin fields. What is missing is an understanding of a generalization of Riemannian geometry which would allow us to write down expressions which are covariant w.r.t. all generalized higher-spin diffeomorphisms, e.g., generalized curvatures.

Even though the Chern-Simons formulation provides a complete description of the system, there are situations where a reformulation in terms of metric-like fields seems desirable. For instance, in the context of holography, the boundary conformal field theory is coupled to the boundary values of the metric-like fields in the bulk. The way we bypass this difficulty in this paper is to use pure gravity as a guiding principle, in the following sense. For pure gravity, both the metric and Chern-Simons formulations are well understood; therefore we can translate all well-known features, in particular those which are relevant in the context of holography, from the metric formulation to the connection one and then look for a natural generalization to higher-rank gauge groups, as appropriate for the description of higher-spin fields.

One of the earliest results in the AdS/CFT correspondence is the holographic description (in any even dimension) of the Weyl anomaly in terms of the dual bulk gravity theory, which plays the role of the non-local effective action [?]. For a three-dimensional bulk, one can translate the analysis into the equivalent Chern-Simons formulation and interpret the bulk diffeomorphism, which induces a Weyl rescaling of the boundary metric, as a particular gauge transformation. In the same way as the Weyl anomaly can be interpreted as the non-invariance of the bulk action under bulk diffeomorphism due to the presence of a boundary, it can be alternatively interpreted as the non-invariance of the Chern-Simons action under gauge transformations, again due to the appearance of a boundary term. Once this is realized, a generalization to higher-rank gauge groups, i.e., to higher-spin theories, is possible, in the sense that W-Weyl symmetries can be interpreted as particular gauge transformations and the non-invariance of the Chern-Simons action can be interpreted as the non-invariance of the non-local effective action of the boundary theory, thus representing the anomalies. The relevant gauge transformations turn out to be those generated by the diagonal Cartan subalgebra of the two $sl(N)$ factors.

The outline of the paper is as follows. In the second chapter we reformulate many features of pure three-dimensional AdS-gravity in the language of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory. This is mostly a review of well-known facts and follows to a large extent [?], in particular in translating the Fefferman-Graham gauge for the metric to the connection formulation of the theory. We then consider different gauge choices for the connection which correspond to different boundary metrics for the dual CFT. In the third chapter we extend the analysis to higher-rank Chern-Simons theories. We make an attempt to reinterpret the gauge theory results in terms of the higher-spin metric-like fields.

As the main new features (and difficulties) arise already for $SL(3)$, we will restrict mostly to this case, i.e., to spin three. We define the Fefferman-Graham gauge and among the residual gauge transformations those which induce W-Weyl rescaling of the boundary fields. We use them to compute the variation of the Chern-Simons action, which is interpreted as the effective action of the boundary theory. We do this in the same gauges which we studied in Chapter 2. They were also studied recently, however with different emphasis, in [?] and [?], respectively. The interpretation of our result, which also touches upon the interpretation of the relation between bulk and boundary fields, does not seem to be straightforward, though. In Appendix A we establish our conventions for the $sl(N)$ algebras and their representations. In Appendix B we collect some results for general $sl(N)$.

2.1 Generalities

Our objectives are higher spin theories. As a preparation we review the CS-formulation of pure three-dimensional gravity and state some of the relevant features in a way that suggests a natural generalization to the higher-spin case.

The action of three-dimensional gravity with a cosmological constant in the second-order formulation is

$$S_{EH} = \frac{1}{16\pi G_N}\int_M d^3x\sqrt{G}\left(R+\frac{2}{\ell^2}\right)$$

where $G_{\mu\nu}$ is the metric and R is the Ricci scalar. ℓ is a length scale, which we will often set to one, and G_N is Newton's constant in three dimensions. The action in the first-order formulation is

$$S_{FO} = \frac{1}{8\pi G_N}\int_M \mathrm{tr}\left(e\wedge R+\frac{1}{6\ell^2}e\wedge e\wedge e\right)$$

where $e = e^a_\mu J_a dx^\mu$ is the $so(2,1)$-valued dreibein, $R = d\omega+\omega\wedge\omega$ the curvature 2-form, and $\omega = \omega^{bc}J_{bc}$ the spin connection. For further details on the notation we refer to Appendix A. The equations of motion for ω are the vanishing of the torsion, which allows us to solve algebraically $\omega = \omega(e)$. The equations of motion for e are then the Einstein equations for the metric.

These formulations of three-dimensional gravity can be trivially generalized to arbitrary dimensions. There is, however, an alternative formulation which does not generalize to higher dimensions, namely in terms of an $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ Chern-Simons theory [?, ?]. If we denote the gauge fields of the two $SL(2)$ factors by A and $\tilde{A}$, respectively, the action (2.2) can be written as

$$S = S_{CS}[A]-S_{CS}[\tilde{A}]+S_{\mathrm{bndy}}$$

where the Chern-Simons actions are

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)$$

and we need to identify

$$k = \frac{\ell}{4G_N}, \quad A + \tilde{A} = \omega, \quad A - \tilde{A} = \frac{1}{\ell} e$$

The difference between the Chern-Simons action and the Einstein-Hilbert action is a boundary term

$$S_{\text{bndy}} = -\frac{k}{4\pi} \int_{\partial M} \text{tr}(\omega \wedge e)$$

The metric can be recovered from the connection via

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{\ell^2}{2} \text{tr} \left[(A - \tilde{A})_\mu (A - \tilde{A})_\nu \right] dx^\mu dx^\nu$$

The equations of motion are the flatness conditions for A and $\tilde{A}$

$$F = dA + A \wedge A = 0, \quad \tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A} = 0$$

They are invariant under $SL(2, \mathbb{R})$ gauge transformations

$$A \to U^{-1} A U + U^{-1} dU, \quad \tilde{A} \to \tilde{U}^{-1} \tilde{A} \tilde{U} + \tilde{U}^{-1} d\tilde{U}$$

whose infinitesimal version is $\delta A = d\lambda + [A, \lambda]$. Here $U = \exp(\lambda)$ with $\lambda \in sl(2, \mathbb{R})$.

If we define

$$\zeta = \frac{1}{2}(\lambda + \tilde{\lambda}), \quad \Lambda = \frac{1}{2}(\lambda - \tilde{\lambda})$$

then the infinitesimal version of (2.9) gives

$$\delta_\zeta e = d\zeta + [\omega, \zeta], \quad \delta_\Lambda e = [e, \Lambda]$$

Comparing this with pure gravity [**?**] identifies ζ as the parameters of diffeomorphisms and Λ as those of Lorentz transformations. In these expressions ζ is Lie algebra valued. The corresponding space-time vector ξ is

$$\xi_\mu dx^\mu = \frac{1}{2} \text{tr}(e_\mu \zeta) dx^\mu$$

Holographic considerations usually use the metric formulation based on the Einstein-Hilbert action. To this end, the Fefferman-Graham (FG) gauge for the metric, Fefferman-Graham expansion, and the Penrose-Brown-Henneaux (PBH) transformations have proven very useful. One uses diffeomorphisms to bring the metric to the FG form

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{\ell^2}{\rho^2} \left(d\rho^2 + g_{ij}(\rho, x^i) dx^i dx^j \right)$$

where ρ is the radial coordinate and $\rho = 0$ is the boundary with coordinates x^i. This gauge is particularly convenient in writing the bulk/boundary dictionary as there are no cross-terms $G_{\rho i}$. As shown in [?], $g_{ij}(\rho,x)$ has an expansion in the vicinity of the boundary (FG expansion)

$$g_{ij}(\rho,x)=\sum_{n=0}^{\infty}\rho^{2n}g_{ij}^{(2n)}(x)$$

where $g_{ij}^{(0)}$ is the boundary metric. For even boundary dimension d there are additional terms containing logarithms of the radial coordinate. In $d=2$, which is the case we are interested in, they are however absent. Furthermore, in $d=2$, in contrast to higher dimensions, the FG expansion is finite [?, ?]. It terminates after the third term $\rho^2 g_{ij}^{(4)}(x)$, which is completely fixed in terms of the lower terms as

$$g_{ij}^{(4)}=\frac{1}{4}g_{ik}^{(0)}g_{jl}^{(0)}T^{kl}$$

where T^{kl} is the vacuum expectation value of the conserved stress-energy tensor of the boundary CFT coupled to an external metric $g_{ij}^{(0)}$ [?]

$$\langle T_{ij}\rangle=\frac{2}{\sqrt{g^{(0)}}}\frac{\delta W[g^{(0)}]}{\delta g^{(0)ij}}=\frac{k}{\pi}\left(t_{ij}+q_{ij}\right)$$

where t_{ij} is a (non-local) functional of $g_{ij}^{(0)}$ and q_{ij} are the boundary data which specify a bulk solution. q_{ij} is traceless and conserved w.r.t. $g_{ij}^{(0)}$. The FG gauge is not a complete gauge fixing. The residual diffeomorphism, called PBH transformations, are generated by those ξ^μ which satisfy

$$\mathcal{L}_\xi G_{\rho i}=0,\quad \mathcal{L}_\xi G_{\rho\rho}=0$$

whose solution is [?]

$$\xi^\rho(\rho,x)=\rho\sigma(x),\quad \xi^i(\rho,x)=\partial_j\sigma(x)\int_0^\rho d\rho'\rho' g^{ij}(\rho',x)+\xi^i(0,x)$$

Except for the boundary term $\xi^i(0,x)$, which generates an uninteresting boundary diffeomorphism and which will be set to zero from here on, the PBH transformation is parametrized by a single function $\sigma(x)$ on the boundary. A PBH transformation acts on $g_{ij}(\rho,x)$ as

$$\delta_\xi g_{ij}=\sigma(2\rho\partial_\rho)g_{ij}+\nabla_i\xi_j+\nabla_j\xi_i$$

where ∇ is w.r.t. g_{ij} and $\xi_i\equiv g_{ij}\xi^j$. This implies

$$\delta_\xi g_{ij}^{(0)}=2\sigma g_{ij}^{(0)}$$

i.e., the bulk PBH transformation induces a Weyl rescaling of the boundary metric which integrates to $e^{2\sigma(x)}g_{ij}^{(0)}$ for finite transformations. It is easy to work out the PBH transformations of the higher $g^{(2n)}$; for instance,

$$\delta_\xi g_{ij}^{(2)}=\nabla_i\nabla_j\sigma$$

which is solved by

$$g_{ij}^{(2)} = \frac{1}{2}\left(R_{ij} - \frac{1}{4}g_{ij}\mathrm{tr}(g^{(0)-1}R)\right)$$

and the fact that $g_{ij}^{(4)}$ is finite reflects the locality of the Weyl anomaly. It does not yet contain the second set of boundary data, q_{ij}. In addition, if other fields are present which allow for the construction of Weyl-invariant symmetric tensors, they can also contribute to $g_{ij}^{(4)}$. To fix them we need to go on-shell.

Using holography there is an easy way to compute the Weyl anomaly of the boundary CFT, i.e., the non-invariance of the effective action $W[g]$ under Weyl rescaling of g. $W[g]$, the generating function for correlation functions of the energy-momentum tensor, is obtained by coupling the CFT to an external metric g and integrating out the CFT. A bulk diffeomorphism leaves the dual gravity action with Lagrangian $\mathcal{L}$ invariant, up to a boundary term

$$\delta_\xi S = \int_{\partial M} \partial_\mu(\xi^\mu \mathcal{L})$$

If we go to FG gauge, perform a FG expansion of the integrand, and use a PBH diffeomorphism, i.e., $\xi^\rho = \rho\sigma$, the on-shell ρ^0 term is the anomaly [?]

$$\delta_\sigma W[g] = \delta_\xi S|_{\text{on-shell}}$$

Possible divergencies at $\rho = 0$ are cancelled by adding local boundary terms to the bulk action. Applied to (2.1) this gives

$$\delta_\sigma W[g] = \frac{c}{24\pi}\int d^2x\sqrt{g}\sigma R$$

where R is the Ricci scalar of the boundary metric.

We will now translate these results to the CS formulation (2.3). Here, of course, we will have to set $d = 2$.

In the coordinates $x^\mu = (\rho, x^1, x^2)$ the connection decomposes as

$$A = A_\mu dx^\mu = A_\rho d\rho + A_i dx^i, \quad A_\mu \in sl(2,\mathbb{R})$$

Using the invariance of the action under (2.9) we can choose a gauge for $(A, \tilde{A})$ that best suits the holographic description: the analogue of the Fefferman-Graham gauge with $G_{\rho\rho} = 1/\rho^2$ and $G_{i\rho} = 0$. It is easy to see that with

$$A_\rho = \frac{1}{\rho}L_0, \quad \tilde{A}_\rho = -\frac{1}{\rho}L_0$$

(2.7) leads to $G_{\rho\rho} = \mathrm{tr}[(L_0)^2]/\rho^2$. In [?] it was shown that this gauge choice can always be achieved with a group element U that goes to the identity at the boundary. This condition is necessary if we want that any Dirichlet boundary condition (in [?] it was $A_\rho = 0$ at $\rho = 0$) is preserved.

Our gauge choice for $(A, \tilde{A})$ is therefore

$$A = b^{-1}ab + b^{-1}db, \quad \tilde{A} = \tilde{b}^{-1}\tilde{a}\tilde{b} + \tilde{b}^{-1}d\tilde{b}$$

with

$$b = e^{(\log\rho)L_0}, \quad \tilde{b} = e^{-(\log\rho)L_0}$$

where $(a, \tilde{a})$ are $sl(2,\mathbb{R})$-valued one-forms along the boundary directions: $a = a_i dx^i$ and $\tilde{a} = \tilde{a}_i dx^i$. With this choice $F_{\rho i} = 0$ leads to

$$\partial_\rho a_i = 0$$

i.e., a depends only on the boundary coordinates x^i. The remaining flatness conditions $F_{ij} = 0$ are simply flatness of a and $\tilde{a}$:

$$da + a \wedge a = 0, \quad d\tilde{a} + \tilde{a} \wedge \tilde{a} = 0$$

All information is now encoded in the connections $(a, \tilde{a})$, which only depend on the boundary coordinates x^i. A generic $a \in sl(2)$ can be expanded as

$$a(x) = a_i(x)dx^i = a_i^+ L_1 + a_i^0 L_0 + a_i^- L_{-1}$$

and, using (2.28),

$$A_i = \rho a_i^+ L_1 + a_i^0 L_0 + \rho^{-1} a_i^- L_{-1}, \quad \tilde{A}_i = \rho^{-1}\tilde{a}_i^+ L_1 + \tilde{a}_i^0 L_0 + \rho\tilde{a}_i^- L_{-1}$$

For the dreibein $e = e_\rho d\rho + e_i dx^i$ defined in (2.5) we obtain

$$e_\rho = \frac{1}{\rho}(a_i^0 - \tilde{a}_i^0)L_0, \quad e_i = \frac{1}{2}\left[(\rho a_i^+ + \rho^{-1}\tilde{a}_i^-)L_1 + (a_i^0 + \tilde{a}_i^0)L_0 + (\rho^{-1}a_i^- + \rho\tilde{a}_i^+)L_{-1}\right]$$

With (2.7) it is clear that the metric will not be in FG gauge, the culprit being the zero mode component in e_i, which leads to $G_{i\rho} \neq 0$. To remove it we use the residual gauge freedom which preserves the gauge choice (2.26). Making a Gauss decomposition of $U(\rho, x)$ and $\tilde{U}(\rho, x)$ (with $\alpha_\pm = \alpha_\pm(x)$, etc.)

$$U = e^{\rho\alpha_+ L_1} e^{\alpha L_0} e^{\rho\alpha_- L_{-1}}, \quad \tilde{U} = e^{\rho\tilde{\alpha}_- L_{-1}} e^{\tilde{\alpha} L_0} e^{\rho\tilde{\alpha}_+ L_1}$$

with the choice $\alpha_+ = \tilde{\alpha}_- = \alpha = \tilde{\alpha} = 0$, leads to [**?**]

$$a_i^0 = \tilde{a}_i^0 = 0$$

and therefore removes the zero modes of the dreibein, giving $G_{i\rho} = 0$. The gauge choices (2.28) and (2.36) are the FG gauge condition in the CS formulation of three-dimensional gravity. We note that the finiteness of the FG expansion of the dreibein and the metric is manifest. One can show that the FG expansions of ξ^i and $\sqrt{G}$ (but not of ξ_i) are finite as well.

With the above gauge choice, the bulk metric (2.7) becomes

$$G_{\rho\rho} = \frac{1}{\rho^2}, \quad G_{i\rho} = 0, \quad G_{ij} = \frac{1}{2}\mathrm{tr}\left[(a_i - \tilde{a}_i)(a_j - \tilde{a}_j)\right]$$

which expands as

$$G_{ij} = \frac{1}{\rho^2} g_{ij}^{(-2)} + g_{ij}^{(0)} + \rho^2 g_{ij}^{(2)}$$

with

$$g_{ij}^{(-2)} = \frac{1}{2}\mathrm{tr}(L_1^2) a_i^+ \tilde{a}_j^+ dx^i dx^j, \quad g_{ij}^{(0)} = \frac{1}{2}\mathrm{tr}(L_0^2)(a_i^+ \tilde{a}_j^- + \tilde{a}_i^+ a_j^-) dx^i dx^j, \quad g_{ij}^{(2)} = \frac{1}{2}\mathrm{tr}(L_{-1}^2) a_i^- \tilde{a}_j^- dx^i dx^j$$

where the coefficients satisfy the flatness condition (2.30). Using those and the FG gauge condition (2.36), one verifies (2.15) and (2.21).

We know from the metric formulation that the gauge fixing is not yet complete. Indeed, transformations parametrized by α and $\tilde{\alpha}$ have a simple effect on $a_i^\pm$ and $\tilde{a}_i^\pm$:

$$a_i^\pm \to e^{\pm\alpha} a_i^\pm, \quad \tilde{a}_i^\pm \to e^{\pm\tilde{\alpha}} \tilde{a}_i^\pm$$

If we define

$$\sigma = \frac{1}{2}(\alpha + \tilde{\alpha}), \quad \tau = \frac{1}{2}(\alpha - \tilde{\alpha})$$

then σ acts as a Weyl rescaling and τ as a Lorentz transformation of the boundary zweibein. In particular

$$g_{ij}^{(0)} \to e^{2\sigma(x)} g_{ij}^{(0)}$$

Of course, the transformation (2.39) reintroduces $G_{i\rho}$ components, but from the previous discussion we know that we can transform them away without affecting the boundary zweibein. In the metric formulation these are the transformations generated by the ξ^i.

We therefore conclude that the transformations parametrized by σ are the PBH transformations of the metric formulation. The remaining two parameters α_- and $\tilde{\alpha}_-$ parametrize boundary diffeomorphisms.

We now discuss the holographic computation of the Weyl anomaly in the CS formulation. For this we apply the procedure outlined above to the action (2.3). On-shell a diffeomorphism of A can be written as a gauge transformation, $\mathcal{L}_\xi A = d\lambda + [A, \lambda] = \delta_\lambda A$ with $\lambda = \imath_\xi A$, and likewise $\delta_\xi \tilde{A} = \delta_{\tilde{\lambda}} \tilde{A}$ with $\tilde{\lambda} = \imath_\xi \tilde{A}$. Under such a transformation with ξ^μ being the PBH diffeomorphism, the action changes as

$$\delta_\sigma W = \frac{k}{2\pi} \int_{\partial M} \sigma \mathrm{tr}\left[L_0(dA + d\tilde{A})\right]$$

Bulk and boundary term in (2.3) give equal contributions. This was also observed in [**?**]. Using (2.38) and the on-shell and FG gauge conditions, one finds

$$\delta_\sigma W[g] = \frac{k}{2\pi} \int d^2x \sqrt{g} \sigma R$$

Comparing (2.44) with (2.23) and (2.24) we verify the known relation $c = 6k$. We note that (2.43) is nothing but the (ρ^0) term of the change of the action under a gauge transformation with parameter $\lambda = \sigma L_0 = \tilde{\lambda}$. This was expected from the discussion above, where we found the relation between PBH transformations and $sl(2)$ gauge transformations.

So far the discussion has been completely general. We will now consider special choices of the boundary metric and translate them into the CS formulation. This discussion follows largely [?].

2.2 Conformal Gauge

The first case is the conformal gauge, where the boundary metric is $g^{(0)}_{ij} = e^{\Phi}\delta_{ij}$. In this case, the non-local Polyakov action $W[g]$, which is completely fixed up to a multiplicative constant c (the central charge of the CFT), becomes the (local) Liouville action for Φ and correlation functions of the energy-momentum tensor are expressed in terms of the Liouville field Φ. In this gauge $g^{(2)}_{ij}$ in (2.20) has a well-defined limit in $d = 2$:

$$g^{(2)}_{ij} = \frac{1}{2}\partial_i\partial_j\Phi - \frac{1}{4}\partial_i\Phi\partial_j\Phi + \frac{1}{8}\delta_{ij}(\partial\Phi)^2 + q_{ij}$$

where we have added a conserved and traceless q_{ij}. Via (2.15) the bulk metric is now completely fixed in terms of Φ and q. Choosing a complex structure on the boundary such that $ds^2 = e^{\Phi}dzd\bar{z}$, one finds for T_{ij} of (2.16)

$$t_{zz} = \frac{1}{4}(\partial\Phi)^2 + \frac{1}{2}\partial^2\Phi, \quad t_{\bar{z}\bar{z}} = \frac{1}{4}(\bar{\partial}\Phi)^2 + \frac{1}{2}\bar{\partial}^2\Phi, \quad t_{z\bar{z}} = \frac{1}{2}\partial\bar{\partial}\Phi$$

We recognize t_{zz} and $t_{\bar{z}\bar{z}}$ as the (traceless) energy-momentum tensor of Liouville theory. The ambiguity in T_{ij}, previously denoted by q, is

$$q_{zz} = q(z), \quad q_{\bar{z}\bar{z}} = \bar{q}(\bar{z}), \quad q_{z\bar{z}} = 0$$

The bulk metric with $\Phi \neq 0$ can be obtained from the one with $\Phi = 0$ by a finite PBH transformation. As we will now show, this can be easily translated to the CS-formulation of pure gravity (and generalized to the higher-spin case, cf. Section 3).

In FG gauge, the flat connections $(a, \tilde{a})$ that correspond to the on-shell bulk metric in conformal gauge are

$$a_z = e^{\phi}L_1 - \frac{1}{4}T_{zz}L_{-1}, \quad a_{\bar{z}} = \bar{\partial}\phi L_0 + \frac{1}{4}\bar{T}_{\bar{z}\bar{z}}L_1$$

$$\tilde{a}_z = \partial\tilde{\phi}L_0 + \frac{1}{4}T_{zz}L_{-1}, \quad \tilde{a}_{\bar{z}} = e^{\tilde{\phi}}L_{-1} - \frac{1}{4}\bar{T}_{\bar{z}\bar{z}}L_1$$

where $\partial\bar{\partial}\Phi$ is the boundary Ricci tensor. The two fields ϕ and $\tilde{\phi}$ satisfy

$$\phi + \tilde{\phi} = \Phi$$

Consider the pair of connections

$$a_0 = [L_1 - q(z)L_{-1}]\,dz, \quad \tilde{a}_0 = [L_{-1} - \bar{q}(\bar{z})L_1]\,d\bar{z}$$

It is obviously flat and in FG gauge. It corresponds to a flat boundary metric and vev's $(q(z), \bar{q}(\bar{z}))$ in the absence of the source. The flat connection (2.48) is related to (2.50) via gauge transformations with

$$g = e^{-\frac{1}{2}\partial\Phi L_{-1}} e^{\phi L_0}, \quad \tilde{g} = e^{-\frac{1}{2}\bar{\partial}\Phi L_1} e^{-\tilde{\phi} L_0} = (g^{-1})^\dagger$$

Note that this gauge transformation does not depend on $(q(z), \bar{q}(\bar{z}))$. The bulk metrics derived from (2.48) and (2.50) are related by a finite PBH transformation generated by (2.51). The zero mode parts in $(g, \tilde{g})$ introduce the conformal mode, while the other factors in $(g, \tilde{g})$ restore the FG gauge. As we have already remarked before, this does not modify the leading terms in the FG expansion.

Finally, applying (2.43) to the connections (2.48) we obtain the conformal anomaly

$$\delta_\sigma W = \frac{k}{\pi} \int \sigma \partial \bar{\partial} \Phi \, dz d\bar{z}$$

Using that the Ricci scalar for the conformal metric is $R = -2e^{-\Phi}\partial\bar{\partial}\Phi$, we confirm that this agrees with (2.44). This can be integrated to

$$W = \frac{k}{2\pi} \int \Phi \partial \bar{\partial} \Phi \, dz d\bar{z}$$

if $\delta_\sigma \Phi = 2\sigma$, which is the Weyl rescaling of the boundary metric in conformal gauge.

2.3 μ-Gauge

The second gauge choice which we will discuss is constructed such that the boundary metric is

$$ds^2 = dzd\bar{z} + \mu d\bar{z}^2 + \bar{\mu} dz^2$$

where the Beltrami differential $\mu(z, \bar{z})$ defines the complex structure. The pair $(\mu, \bar{\mu})$ sources the $(T_{zz}, T_{\bar{z}\bar{z}})$ components of the CFT energy-momentum tensor. The most general metric can be written as $e^\Phi |dz + \mu d\bar{z}|^2$ and our gauge fixing amounts to setting the conformal factor to one. We can restore it via a PBH transformation.

We will now construct the $sl(2) \oplus sl(2)$ connection $(a, \tilde{a})$ from which we can construct the bulk metric in FG gauge with (2.53) as boundary metric. Recall that the Beltrami differential μ parametrizes the complex structure on the boundary. Demanding $du = \lambda(dz + \mu d\bar{z})$ requires u to satisfy the Beltrami equation

$$(\bar{\partial} - \mu \partial) u = 0$$

The pair (μ, T) with

$$T = -\frac{1}{2}\{u, z\} = -\frac{1}{2}\left(\frac{u'''}{u'} - \frac{3}{2}\frac{u''^2}{u'^2}\right)$$

defines a projective structure. Given T and μ, the consistency between (2.54) and (2.55) requires them to satisfy the anomalous Virasoro Ward identity

$$\bar{\partial}T = \frac{c}{12}\partial^3\mu + 2(\partial\mu)T + \mu\partial T$$

We recall that the stress energy $T = \frac{6}{c}T_{zz}$ where $k = c/6$; hence (2.55) agrees with the usual Virasoro Ward identity.

The following observation connects this to the $sl(2)$ CS theory [?]. Consider the linear system

$$(dz(\partial + a_z) + d\bar{z}(\bar{\partial} + a_{\bar{z}}))\Psi = 0$$

where a is an $sl(2, \mathbb{C})$ connection with

$$a_z \equiv L_1 - \frac{1}{4}TL_{-1}, \quad a_{\bar{z}} \equiv \mu L_1 + \bar{\omega}L_0 + \beta L_{-1}$$

The holomorphic and anti-holomorphic parts of equation $\Psi = 0$ imply a second-order holomorphic and a first-order mixed one for ψ, respectively:

$$(\partial^2 + T)\psi = 0, \quad (\bar{\partial} - \mu\partial + \bar{\omega})\psi = 0$$

Compatibility between the holomorphic and the anti-holomorphic parts requires a to be flat. This in turn implies two algebraic equations

$$\bar{\omega} = -\frac{1}{2}\partial\mu, \quad \beta = -\frac{1}{4}\mu T$$

and one first-order ODE, which is precisely the Virasoro Ward identity (2.56). It is then straightforward to show that the ratio of two linearly independent solutions of (2.59), i.e.,

$$u = \frac{\psi_1}{\psi_2}$$

is a solution of (2.54) and (2.55). The linear system (2.57) is therefore equivalent to (2.54) and (2.55) [?].

To derive (2.54) and (2.55) from the flat connection a (2.58) we rewrite a as a pure gauge $a = g^{-1}dg$ and make a Gauss decomposition of g:

$$g = e^{f_+L_1}e^{-f_0L_0}e^{f_-L_{-1}}$$

The minus sign in front of f_0 is for later convenience. The condition for $a_z = g^{-1}\partial g$ to be in the highest weight gauge (2.58) gives

$$e^{f_0} = \partial f_+, \quad f_- = -\frac{1}{2}\frac{\partial^2 f_0}{(\partial f_0)^2}$$

Then T can be read off from the L_{-1} direction of a_z:

$$T = \frac{1}{2}\{f_+, z\}$$

and μ can be read off from the L_1 direction of $a_{\bar{z}}$:

$$\mu = \frac{\bar{\partial} f_+}{\partial f_+}$$

T is the Schwarzian derivative of f_+, which satisfies the Beltrami equation (2.54). (T is also the negative of the energy of Liouville theory with $\Phi = f_0$.)

One is now tempted to use the connection (2.58) and its anti-holomorphic counterpart $a^\dagger$ to construct the bulk metric with boundary metric (2.53). However, this fails because the metric would not be in FG gauge. The latter is obvious as the zero modes of a and $\tilde{a}$ are not coupled. This can be cured with a gauge transformation generated by

$$g = e^{-\frac{1}{2}\omega_{FG} L_{-1}}, \quad \tilde{g} = e^{-\frac{1}{2}\bar{\omega}_{FG} L_1}$$

This leads to

$$a_z = L_1 - \frac{1}{4} T L_{-1}, \quad \tilde{a}_z = \bar{\mu} L_{-1} + \omega L_0 + \bar{\beta} L_1$$

$$a_{\bar{z}} = \mu L_1 + \bar{\omega} L_0 + \beta L_{-1}, \quad \tilde{a}_{\bar{z}} = L_{-1} + \bar{\omega} L_0 - \frac{1}{4} \bar{T} L_1$$

where we have chosen $\omega = \omega_{FG} + \bar{\mu}\bar{\omega}_{FG}$. We have also redefined β and dropped the subscript on ω_{FG}. We will call the gauge (2.66) the μ-gauge.

The flatness condition leads to two algebraic equations for ω and β:

$$\bar{\omega} = -\frac{1}{2}\partial\mu, \quad \beta = -\frac{1}{4}\mu T$$

The remaining two equations are the Virasoro Ward identities:

$$\bar{\partial} T = \frac{c}{12}\partial^3\mu + 2(\partial\mu)T + \mu\partial T$$

and its complex conjugate for $\bar{T}$. The first, combined with its anti-holomorphic counterpart, gives

$$\bar{\partial}\omega + \partial\bar{\omega} = 0$$

The last relation following from flatness is the anomalous Virasoro Ward identity (2.56). The shift of T by $-Q^2$ simply undoes the shift of T when going from (2.58) to (2.66). The ambiguity in T, previously denoted by q, is a solution of the non-anomalous Ward identity. The Ward identity obeyed by $\bar{T}$ is the complex conjugate of (2.69).

From (2.66) we can compute the dreibein and the bulk metric, for which we find

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz) + [(T dz + \beta d\bar{z})(dz + \mu d\bar{z}) + \text{c.c.}]$$

with β as in (2.67).

As a check we compute, using (2.43),

$$\delta_\sigma W[g] = \frac{k}{\pi}\int \sigma(\partial\bar{\omega} + \bar{\partial}\omega)d^2z$$

Using (2.68) one finds $\bar{\partial}\omega + \partial\bar{\omega} = \frac{1}{2}\sqrt{g}R[g]$, which shows that the Weyl anomaly is correctly reproduced.

We can also write down the bulk metric with the boundary metric in lightcone gauge by setting $\bar{\mu} = 0$. This results in the connections

$$a_z = L_1 - \frac{1}{4}TL_{-1}, \quad \tilde{a}_z = \frac{1}{2}\partial\bar{\omega}L_{-1}, \quad a_{\bar{z}} = \mu L_1 + \bar{\omega}L_0 + \beta L_{-1}, \quad \tilde{a}_{\bar{z}} = L_{-1} + \bar{\omega}L_0 - \frac{1}{4}\bar{T}L_1$$

The equation of motion obeyed by $(\bar{\omega}, \beta)$ reduces from (2.67) back to (2.60). Note however that the holomorphic a and anti-holomorphic $\tilde{a}$ are still coupled, so as to ensure FG gauge.

The Ward identity satisfied by T reduces to the chiral one (2.56), whereas the one satisfied by $\bar{T}$ is $\partial(\bar{T} - Q^2(\bar{\omega})) = 0$ with $\bar{\omega} = \partial\mu$. The Weyl anomaly is simply

$$\delta_\sigma W = \frac{c}{24\pi}\int d^2z\sigma\partial^2\mu$$

3.1 Generalities

The goal of the previous chapter was to establish relations between the metric formulation of three-dimensional gravity and its connection formulation as a Chern-Simons theory. In this chapter we will turn our attention to higher-spin theories in three dimensions. Here we have only limited knowledge of its formulation in terms of the metric and higher-spin fields and we thus have to resort to its CS formulation. Much of the discussion of Chapter 2, which was largely review and reformulation of well-known results, was presented in such a way that it can be straightforwardly generalized from $sl(2)$ to $sl(N)$. However, since explicit expressions become rather cumbersome, we will often restrict to $sl(3)$.

The description of higher-spin theories in three dimensions as higher-rank Chern-Simons theories was established in [**?**, **?**, **?**]. The spectrum of spins depends on the embedding of the gravitational $SL(2,\mathbb{R})$ into the gauge group G. Here we will only consider $G = SL(N,\mathbb{R})$ and the principal embedding $SL(2,\mathbb{R}) \hookrightarrow SL(N,\mathbb{R})$.

The starting point for our discussion is the action (2.3) with $A, \tilde{A} \in sl(N,\mathbb{R})$. In order for the gravity subsector to match the Einstein-Hilbert action, we need

$$\frac{k}{2\pi}\mathrm{tr}\left[(L_0)^2\right] = \frac{\ell}{16\pi G_N}$$

where ℓ is the AdS radius, which we will often set to one, and c is the central charge of the boundary CFT. The group theory notation is explained in Appendix A.

The generalized dreibein and spin-connection are again given by (2.5), but now they are elements of $sl(N,\mathbb{R})$. We could also rewrite the action in terms of those fields (see e.g. [?]), but we will instead use (2.3), which is more systematic and elegant. The equations of motion are again the flatness conditions for A and $\tilde{A}$, i.e., $F(A)=0$ and $F(\tilde{A})=0$, and they are invariant under $SL(N)$ gauge transformations.

The parameters ζ and Λ, defined as in (2.10), now parametrize generalized diffeomorphisms and Lorentz transformations. Given ζ, the generators of diffeomorphism and spin-3 transformation are (cf. (2.12))

$$\xi_\mu = \mathrm{tr}(e_\mu \zeta), \quad \xi_{\mu\nu} = \mathrm{tr}(e_{(\mu} e_{\nu)} \zeta)$$

and similarly for spin-4 gauge transformations.

For principal embedding, which is essentially unique, there is one spin-s field $\Psi^{(s)}$ for $s=2,\dots,N$, one for each Casimir invariant; they are totally symmetric rank-s space-time tensors with additional constraints, e.g., double tracelessness in the free theory. They can all be constructed from e. The metric ($s=2$) is

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\mathrm{tr}\,[(L_0)^2]} \mathrm{tr}(e_\mu e_\nu) dx^\mu dx^\nu$$

where the normalization has been chosen to make it independent of the normalization of the generators of $sl(N)$. Similarly, the spin-3 field is the unique (up to a normalization) totally symmetric rank-3 tensor which can be constructed from e:

$$\Psi^{(3)} = \Psi_{\mu\nu\lambda} dx^\mu dx^\nu dx^\lambda = \frac{1}{2} \mathrm{tr}(e_{(\mu} e_\nu e_{\lambda)}) dx^\mu dx^\nu dx^\lambda$$

By construction the higher-spin fields are invariant under generalized Lorentz transformations. For $s>3$ this criterion leaves some ambiguities; e.g., for the $s=4$ field, any linear combination

$$\Psi^{(4)}_{\mu\nu\rho\sigma} = c_1 \mathrm{tr}(e_{(\mu} e_\nu e_\rho e_{\sigma)}) + c_2 \mathrm{tr}(e_{(\mu} e_\nu) \mathrm{tr}(e_\rho e_{\sigma)})$$

is a Lorentz invariant symmetric rank-4 space-time tensor.

The invariance of the action under (2.9) can again be used to choose the gauge (2.26) which, in addition to (2.27), also implies $\Psi^{(3)}_{\rho\rho\rho}=0$. This is necessary if we want to have a pure gravity limit. In fact, if we require $A_\rho = \tilde{A}_\rho$ (i.e., symmetry between the two connections) and that in the pure gravity case (i.e., when we switch off all higher-spin fields) A_ρ and $\tilde{A}_\rho$ reduce to (2.26), this is the only choice. For $s=4$ this requirement fixes the coefficient c in (3.5). We can therefore impose

$$\Psi^{(s)}_{\rho\dots\rho} = 0$$

as part of our FG gauge condition. As shown in [?] this gauge choice is always possible and can be achieved by a group element that goes to the identity at the boundary $\rho = 0$.

Our gauge choice for $(A, \tilde{A})$ is therefore again (2.28) where $(a, \tilde{a})$ are now $sl(N, \mathbb{R})$-valued and $F_{\rho i} = 0$ leads again to (2.29) and (2.30). The mode expansions (2.31) are generalized to

$$a_i = \sum_{s=2}^{N} \sum_{m=-(s-1)}^{s-1} a_{i,m}^{(s)} W_m^{(s)}, \quad \tilde{a}_i = \sum_{s=2}^{N} \sum_{m=-(s-1)}^{s-1} \tilde{a}_{i,m}^{(s)} W_m^{(s)}$$

with $a_{i,m}^{(s)}$ being one-forms on the boundary. Note that the ρ-dependence is completely fixed and the residual gauge transformations, to be discussed next, are parametrized by functions on the boundary.

The choice (2.28) has a residual gauge freedom with

$$U(\rho, x) = b^{-1} u(x) b, \quad \tilde{U}(\rho, x) = \tilde{b}^{-1} \tilde{u}(x) \tilde{b}$$

which acts as

$$a \to u^{-1} a u + u^{-1} du, \quad \tilde{a} \to \tilde{u}^{-1} \tilde{a} \tilde{u} + \tilde{u}^{-1} d\tilde{u}$$

Make Gauss decompositions

$$u = u_+ u_0 u_-, \quad \tilde{u} = \tilde{u}_- \tilde{u}_0 \tilde{u}_+$$

where u_+ is generated by all the positive modes $W_m^{(s)}$ with $m > 0$, etc. The conditions for U and $\tilde{U}$ to go to the identity at the boundary are

$$u_+ = u_0 = 1 = \tilde{u}_0 = \tilde{u}_+$$

For $sl(2)$, $G_{i\rho} = 0$ is equivalent to the condition that e_i has no L_0 component. The generalization to $sl(N)$ is that e_i has no $W_0^{(s)}$ components, i.e.,

$$\operatorname{tr}\left[W_0^{(s)}(a_i - \tilde{a}_i)\right] = 0 \quad \text{with } s = 2, \ldots, N$$

This can be achieved with $u = u_-$ and $\tilde{u} = \tilde{u}_+$ and leads to

$$\Psi_{\rho\ldots\rho i}^{(s)} = 0$$

In pure gravity we could gauge away all mixed components. This is not possible for the higher-spin fields.

Before we continue to compute the FG expansion of the bulk spin-s field, we briefly discuss what we should expect from the boundary point of view. A field $\Phi(x)$ with scaling dimension Δ, when coupled to gravity, has Weyl weight Δ:

$$g_{ij} \to e^{2\sigma(x)} g_{ij}, \quad \Phi_\Delta(x) \to e^{\Delta\sigma(x)} \Phi_\Delta(x)$$

where $\sigma(x)$ is the Weyl factor. In flat space a conserved spin-s current $W_{i_1 \dots i_s}$ has scaling dimension, hence Weyl weight, $\Delta = s$. Coupling $W_{i_1 \dots i_s}$ to the background spin-s field $\phi^{(s)}$ via

$$\int d^d x \sqrt{g} W_{i_1 \dots i_s} \phi^{i_1 \dots i_s}$$

and requiring Weyl invariance of (3.15) fixes the Weyl weight of the source $\phi^{i_1 \cdots i_s}$ to be $\Delta = 2 - s$:

$$\phi^{i_1 \dots i_s} \to e^{(s-2)\sigma} \phi^{i_1 \dots i_s}$$

For the metric ($s = 2$), which is the source for the energy-momentum tensor with $\Delta = d$, this is the usual Weyl rescaling.

In the holographic description, the sources are boundary values of bulk fields. Since the Weyl rescaling of the boundary metric is induced by a bulk diffeomorphism with $\xi^\rho = \rho\sigma(x)$, this diffeomorphism must also lead to a rescaling of the boundary value of the spin-s fields. Given their transformation under Weyl rescalings this means that

$$\Psi^{(s)}_{i_1 \dots i_s}(\rho, x) = \phi^{(s)}_{i_1 \dots i_s}(x) \rho^{s-2} + \dots$$

when all components are along the boundary. For bulk fields with mixed components

$$\Psi^{(s)}_{\rho \dots \rho i_1 \dots i_k}(\rho, x) = \phi^{(s)}_{\rho \dots \rho i_1 \dots i_k}(x) \rho^{s-2k-2} + \dots$$

For pure gravity the FG expansion of the metric can be translated to a FG expansion of the dreibein (or vielbein, in general). In the CS formulation it translates into a ρ-expansion of the connections. In FG-gauge

$$A = b^{-1} a b + b^{-1} db, \quad \tilde{A} = \tilde{b}^{-1} \tilde{a} \tilde{b} + \tilde{b}^{-1} d\tilde{b}$$

The remaining two components have the ρ-expansion

$$e^i_{(n)} dx^i = \rho^n \sum_{s=2}^{N} \sum_{m=-(s-1)}^{s-1} \left(a^{(s)}_{i,m} - \tilde{a}^{(s)}_{i,m} \right) W^{(s)}_m dx^i$$

for a generic $(a, \tilde{a})$ with mode-expansion (3.7).

In terms of the ρ-expansion of e, the Fefferman-Graham gauge (3.12) is that e_i has no (ρ^0) term. The ρ-expansion of the metric-like fields can then simply be computed from their definitions in terms of the dreibein e. The finiteness of the FG expansion is an immediate consequence of the construction, e.g., for the metric

$$G_{ij}(x, \rho) = \sum_{n=-1}^{N-2} \rho^{2n} g^{(2n)}_{ij}(x)$$

where $g^{(-2)}_{ij} = 0$ for $n < 0$. However, the back-reaction due to the higher-spin fields changes the leading behavior of the metric. This is to be expected, since the higher-spin fields at the boundary are irrelevant perturbations of the boundary CFT. We will later discuss special cases where the strong back-reaction is absent. The ρ-expansion of the bulk spin-s field can also be easily worked out using (3.21).

We now discuss residual gauge transformations which preserve FG gauge. As in the pure gravity case, we call them PBH transformations. In pure gravity we saw that they induce Weyl transformations of the boundary metric, which, in the CS formulation, are generated by gauge transformations along the Cartan direction L_0, accompanied by compensating gauge transformations which vanish at the boundary and restore FG gauge. The generalization to $sl(N)$ are the gauge transformations along the Cartan directions $W_0^{(s)}$ accompanied by gauge transformations which vanish at the boundary and restore FG gauge.

Infinitesimal gauge transformations which preserve (2.26) are of the form

$$\delta A = d\lambda + [A, \lambda], \quad \lambda = b^{-1}\alpha(x)b$$

together with the $\tilde{A}$ part; α has the mode expansion

$$\alpha = \sum_{s=2}^{N} \sum_{m=-(s-1)}^{s-1} \alpha_m^{(s)} W_m^{(s)}$$

Demanding this to preserve (3.12) imposes the following constraints on the parameters in $(\alpha, \tilde{\alpha})$:

$$\mathrm{tr}\left[W_0^{(s)}(\delta a_i - \delta\tilde{a}_i)\right] = 0 \quad \text{with } s = 2, \ldots, N$$

The gauge transformation (3.23) contains the (generalized) diffeomorphism δ_ζ and the (generalized) Lorentz transformation δ_Λ (2.11). By construction, the Lorentz transformation has no effect on the metric-like fields because all $\mathrm{tr}[W_0^{(s)} W_m^{(t)}] = 0$ for $m \neq 0$. The effect of δ_ζ on the metric-like fields can be computed straightforwardly:

$$\delta_\zeta G_{\mu\nu} dx^\mu dx^\nu = \mathrm{tr}[(L_0)^2]\left(\mathrm{tr}[d\zeta e] + \mathrm{tr}[[\omega, \zeta]e]\right)$$

$$\delta_\zeta \Psi_{\mu\nu\sigma} dx^\mu dx^\nu dx^\sigma = \frac{3}{2}\left(\mathrm{tr}[d\zeta e] + \mathrm{tr}[[\omega, \zeta]e]\right) \otimes \mathrm{tr}(ee) + \ldots$$

We emphasize that restricting the gauge transformations to lie in the gravitational $sl(2) \oplus sl(2)$ subalgebra does not imply that the corresponding transformation on the metric-like fields is an ordinary diffeomorphism: in (3.2) the spin-3 transformation $\xi_{\mu\nu}$ can be non-vanishing even when ζ lies only in the $sl(2)$ spanned by $L_0, L_{\pm 1}$ (unless we also restrict the dreibein e to the same $sl(2)$). The gauge transformation that corresponds to pure diffeomorphism is simply given by $g_{\mu\nu} e^\mu \mathrm{tr}(e^\nu \zeta)$ (cf. the first equation in (3.2) and eq. (3.18) of [?]).

We now separate ζ into positive, zero and negative powers of ρ as

$$\zeta = \zeta_+ + \zeta_0 + \zeta_-$$

where

$$\zeta_0 = \sum_{s=2}^{N} \sigma_s(x) W_0^{(s)}$$

Generalizing the discussion of the previous chapter we expect that ζ_0 parametrizes Weyl-W-Weyl transformations of the boundary fields and that ζ_+ can be used to transform the connections back to FG gauge. We therefore define

$$\delta_{\sigma_s} \equiv \delta_{\zeta=\sigma_s(x) W_0^{(s)}}$$

as the parameters of W-Weyl transformations.

To make the discussion more concrete, we discuss in d

Note: Figure translations are in progress. See original paper for figures.