Twisted sectors from plane partitions (postprint)

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Full Text

Preamble

Twisted sectors from plane partitions

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Contents

Introduction

Massless higher spin theories can be constructed consistently on AdS backgrounds [1], and they are believed to describe a consistent subsector of string theory at the tensionless point [2–4]. Via the AdS/CFT duality, the tensionless limit of string theory corresponds, on the dual field theory side, to the limit in which the CFT becomes free, with the higher spin degrees of freedom corresponding to those of a vector-like CFT. The AdS/CFT correspondence therefore predicts dualities between higher spin theories and vector-like CFTs, with explicit examples first proposed for AdS_4/CFT_3 in [5, 6] and more recently in one dimension lower in [7].

These dualities provide a useful approach toward analyzing string theory at a very symmetrical point in its moduli space where many of its underlying symmetries are unbroken, and they may also allow one to prove the AdS/CFT correspondence since they are weak-weak dualities. To utilize the duality for either purpose, it is crucial to understand the embedding of the higher spin AdS/CFT duality into the usual stringy AdS/CFT correspondence in detail. For the 4d/3d case, a proposal was made some time ago in [8], while for AdS_3/CFT_2 a somewhat different picture emerged in [9]. The CFT duals of string theory on $AdS_3 \times S^3 \times T^4$ lie on the moduli space that contains the symmetric orbifold of T^4, and the orbifold theory itself contains a vector-like CFT as a closed subsector. This vector-like CFT was shown to emerge naturally in the CFT dual of the $N = 4$ version [10] of the higher spin AdS/CFT duality. More specifically, the CFT dual of the higher spin theory is a subsector of the untwisted sector of the symmetric orbifold, and the entire untwisted sector can be understood in terms of a vastly extended higher spin symmetry, the so-called Higher Spin Square (HSS), together with its scalar field excitations [11, 12] (see also [13, 14] for related discussions). However, characterizing the twisted sector of the symmetric orbifold from a higher spin perspective has proven more difficult—see [12, 15, 16] for first steps in this direction. This remains an important problem if we want to use the higher spin perspective to describe string theory at its highly symmetrical tensionless point.

The present work revisits the original bosonic duality of [7] to analyze the relevant twisted sectors from a higher spin perspective. The bosonic theory serves as a useful toy model since it exhibits all the essential features of the supersymmetric version. While the identification of twisted sectors in terms of coset representations was previously accomplished for the $N = 2$ and $N = 4$ cases [9, 17], partially using the BPS condition as a guide, the bosonic case has not yet been worked out. The structure of the bosonic coset differs somewhat from its supersymmetric counterparts, making direct generalization unclear.

In this paper, we attack this problem using a new tool recently discovered for bosonic W∞ algebras. A few years ago, a series of papers [18-20] showed that the representation theory of the quantum toroidal algebra of gl_1 can be described in terms of plane partitions and that the associated characters are, up to an overall q-Pochhammer symbol, identical to those of the bosonic W_{N,k} minimal models. More recently, Procházka [21] realized this provides a powerful method to analyze bosonic W∞ representations, demonstrating that the triality symmetry of the W∞ algebra—which played an important role in establishing the duality [22]—is inherent and manifest in the plane partition description. Since quantum toroidal algebras are isomorphic to their corresponding affine Yangian algebras (after suitable completion) [23, 24], plane partitions also describe the representation theory of a Yangian algebra. Yangian algebras are hallmarks of integrability, and hence this viewpoint may help establish the precise connection between higher spin theories and integrable field theories proposed for AdS_3 in [25-27] (see [28] for a review).

The $N = 4$ construction of [9] relates Wolf space cosets to the symmetric orbifold (and hence to string theory) for cases where the cosets can be described in terms of free fields ($\lambda = 0$). Free field constructions of the bosonic W∞[λ] algebras arise for $\lambda = 0$ and $\lambda = 1$, where they can be realized in terms of free fermions and bosons, respectively [29-33]. For $\lambda = 0$, the free field realization of the coset model was already made fairly explicit in [34], where the $k \to \infty$ (i.e., $\lambda \to 0$) limit was described as a continuous orbifold. However, the precise description of the twisted sectors was not understood at the time—in general, the twisted sector of an orbifold is not directly accessible even with a good understanding of the untwisted sector. In this paper, we find the coset description of the twisted sectors at both the free fermion ($\lambda = 0$) and free boson point ($\lambda = 1$). Controlling both cases simultaneously is important since the extended higher spin symmetry algebra believed to arise in string theory, the Higher Spin Square (HSS), is in some sense a combination of both constructions [11]. The main technical advance of our analysis is the systematic use of the plane partition viewpoint advocated in [21], which enabled us to find the correct twisted sector representations. We also test our proposals using the techniques of [17].

The paper is organized as follows. Section 2 discusses the bosonic theory corresponding to $\lambda = 1$ (i.e., $N \to \infty$ at fixed k). We first find closed-form expressions for the wedge characters of the twisted sectors and then use the plane partition viewpoint to propose the form of the corresponding coset representations. This proposal is tested in detail: in Section 2.3, the null-vector structure of the corresponding hs[λ] representations is studied from a microscopic viewpoint by calculating the relevant Kac determinants, and in Section 2.4, the conformal dimension and excitation spectrum is computed in the coset and found to agree with the orbifold predictions, which also fixes the precise identification with the coset representations. Section 3 performs the corresponding analysis for the fermionic theory corresponding to $\lambda = 0$. We conclude with a discussion of future directions in Section 4. Three appendices explain aspects of the null-vector analysis (Appendix A), the determination of higher spin charges using the Drinfeld-Sokolov approach (Appendix B), and combinatorial identities that arise in the plane partition analysis (Appendix C).

2 The twisted sector in the free boson description

We are interested in the cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$$

and we shall mainly consider the ' t Hooft limit, where we take N and k to infinity while keeping the ratio

$$\lambda = \frac{N}{N+k}$$

fixed. The case where we take N $\to \infty$ first corresponds to the theory at $\lambda = 1$.

This limit theory can be described by a free boson construction (see e.g., [32, 33]). More specifically, for k complex bosons ϕ and $\bar{\phi}$ that transform in the fundamental and anti-fundamental representation of U(k), respectively, we consider the chiral U(k) singlets of the form

$$W^s(z) = m(s) \sum_{l=0}^{s-1} (-1)^l \binom{s-1}{l} \binom{s-1}{s-l} \partial^l \phi_j \partial^{s-l} \bar{\phi}^j,$$

where m(s) is an s-dependent normalization constant. These currents generate the W∞[1] algebra with c = 2k. Formally, the $\lambda = 1$ theory can therefore be thought of as a continuous orbifold where we divide the free boson theory by the orbifold group U(k) (see [34]). The theory should then also contain twisted sectors where the different complex bosons are twisted.

Note that some of these twisted sectors also appear in the symmetric orbifold where we divide the theory by the symmetric group S_{k+1} U(k), under which the above currents are also invariant. (The full chiral algebra of the symmetric orbifold is then much bigger—it gives rise to the stringy extension of the W∞[1] algebra associated with the so-called Higher Spin Square [11].) The analysis of the twisted sectors is therefore particularly relevant for the stringy embedding of the bosonic duality of [7].

In general, the coset interpretation of the twisted sectors cannot be deduced directly from the identification in the untwisted sector, so we shall proceed indirectly. Let us first concentrate on the case where only one complex boson is twisted by with 0 < < 1; the twisted sector is then generated by

$$\bar{\alpha}_{n+\nu} \quad \text{with} \quad n \in \mathbb{Z}.$$

The wedge character of the corresponding representation is given by [12]

$$\chi^{[\nu]}(q,y) = q^h \prod_{n=1}^{\infty} (1 - yq^{n-1+\nu})^{-1}(1 - y^{-1}q^{n-\nu})^{-1},$$

where the conformal dimension is h = ½ (1-). In the full orbifold theory, this chiral representation comes together with a corresponding anti-chiral representation, and on the full space (involving both chiral and anti-chiral twisted states) the invariance under the orbifold group must be imposed. In particular, not only those states that are separately invariant under the orbifold action survive this orbifold projection; instead, the correct condition is that the left-moving states transform in the conjugate representation to that of the right-moving states. The powers of y keep track of the action under the cyclic group corresponding to the twist itself, and hence the states corresponding to a given fixed power of y correspond to different representations of the hs[1] algebra,

$$\chi^{[\nu]}_{(\ell)}(q) \equiv \chi^{[\nu]}(q,y)\big|_{y^\ell}.$$

In the following, we identify the coset representations that describe these twisted sector states. We first use character considerations to propose the corresponding coset representation (see Sections 2.1 and 2.2). In Section 2.3, we study the representation corresponding to = 0 using the commutation relations of the hs[λ] algebra and confirm that the representation identified in Section 2.1 leads to the correct eigenvalues for arbitrary λ. (As will become clear, these representations can also be defined for general λ; however, we do not have a direct interpretation in terms of an orbifold unless $\lambda = 1$.) Finally, in Section 2.4, we confirm that the representations have the correct ground state conformal dimension and excitation spectrum.

2.1 Wedge characters and their combinatorial interpretation

To use the characters for determining the corresponding coset representations, we first compute the wedge characters defined in eq. (2.6). In particular, we want to find closed-form expressions whose combinatorial interpretation can help us identify their corresponding plane partition configurations.

There are two factors in the wedge character (2.5), corresponding to modes associated with ϕ and $\bar{\phi}$, respectively. Both resemble the refined version of the generating function of partition numbers defined by

$$Z(q,y) = \prod_{n=1}^{\infty} (1 - yq^n)^{-1} = \sum_{n,m} p(n,m) q^n y^m,$$

where p(n, m) counts the number of Young diagrams with n boxes and height m. Summing over all Young diagrams with fixed height m gives

$$\sum_{n} p(n,m)q^n = q^m \prod_{n=1}^{m} (1-q^n)^{-1}.$$

Expanding both factors in the wedge character ^{[ ]}(q, y) as in (2.7) and (2.8) and collecting the coefficient of the y^ term, we obtain the expression for the wedge characters

$$\chi^{[\nu]}_{(\ell)}(q) = q^{h+\delta h(\ell,\nu)} \sum_{m=0}^{\infty} \frac{q^{m+|\ell|}}{\prod_{n=1}^{m+|\ell|}(1-q^n)} \prod_{n=1}^{m} (1-q^n),$$

where

$$\delta h(\ell,\nu) = \begin{cases} \ell\nu & \ell \geq 0 \\ \ell(\nu-1) & \ell < 0 \end{cases}$$

corresponds to the excitation spectrum, and   enters the combinatorial part of the character only as | |. The explicit q-expansions for the first few values of | | are:

$$\chi^{[\nu]}_{(0)}(q) = q^h \left(1 + q + 3q^2 + 6q^3 + 12q^4 + 21q^5 + 38q^6 + 63q^7 + \cdots\right),$$

$$\chi^{[\nu]}_{(1)}(q) = q^{h+\nu} \left(1 + 2q + 4q^2 + 8q^3 + 15q^4 + 27q^5 + 47q^6 + 79q^7 + \cdots\right),$$

$$\chi^{[\nu]}_{(2)}(q) = q^{h+2\nu} \left(1 + 2q + 5q^2 + 9q^3 + 18q^4 + 31q^5 + 55q^6 + 91q^7 + \cdots\right),$$

$$\chi^{[\nu]}_{(3)}(q) = q^{h+3\nu} \left(1 + 2q + 5q^2 + 10q^3 + 19q^4 + 34q^5 + 60q^6 + 100q^7 + \cdots\right).$$

For the   = 0 representation, there is one descendant at level one, while for   ≠ 0, the representation has two descendants at level one. This property is directly visible in (2.9) since the first product only contributes a state at level 1 if | | > 0.

Analogous to the counting in eq. (2.8), the wedge character ^{[ ]}_{( )}(q) has a combinatorial interpretation

$$\chi^{[\nu]}_{(\ell)}(q) = q^{h+\delta h(\ell,\nu)} \sum_n p_2(n,\ell) q^n,$$

where p_2(n,) counts pairs of Young diagrams Γ± whose height difference is , i.e., $c^+{}_1$ - $c^-{}_1$ = . Here c± are the number of boxes in the i-th row of Γ±, and n is the combined number of boxes in the two Young diagrams, except that for the first column of each diagram, only the boxes of the shorter diagram are counted:

$$n = \min(c_1^+, c_1^-) + \sum_{i\geq 2} (c_i^+ + c_i^-).$$

The reason for this unusual condition is that in the first factor of ^{[ ]}(q, y), the prefactor is yq^{-1} (instead of y). A useful way to visualize this configuration is to first raise the shorter Young diagram (with height m) to the same height as the taller one (with height m + | |), then glue the two Young diagrams together along their first columns, and finally remove the | | boxes in the first column of the taller Young diagram that are not covered by the shorter diagram (see Figure 1 [Figure 1: see original paper]).

There is an alternative formula for the wedge character that makes the connection to the original complex boson more transparent:

$$\chi^{[\nu]}_{(\ell)}(q) = q^{h+\delta h(\ell,\nu)} \left(\sum_{m=0}^{\infty} (-1)^m q^{\sum_{k=|\ell|+1}^{|\ell|+m} k} \right) \prod_{n=1}^{\infty} (1-q^n)^2.$$

This can be obtained from eq. (2.12) using the combinatorial identity

$$p_2(n,\ell) = \sum_{m=0}^{\infty} (-1)^m p_2 \left(n - \sum_{k=|\ell|+1}^{|\ell|+m} k \right),$$

where p_2(n) counts pairs of Young diagrams whose total number of boxes is n via the generating function of the complex boson

$$Z_{\text{cplx bos}}(q) = \prod_{n=1}^{\infty} (1-q^n)^{-2} = \sum_n p_2(n) q^n = 1+2q+5q^2+10q^3+20q^4+36q^5+65q^6+110q^7+\cdots.$$

A proof of (2.15) for   = 0 is given in [35]; we provide the generalization to arbitrary   in Appendix C. In this formula, the property that the wedge character has only a single descendant at level one for   = 0 comes from the fact that the first factor starts with 1 - q^{| |+1} + , i.e., it removes a state at level one for   = 0 but not otherwise.

As an aside, we mention that the asymptotics of the two-partition function $p_2(n)$ is [36]

$$p_2(n) \sim \frac{1}{4\sqrt{3}n^{5/4}} \exp\left(2\pi\sqrt{\frac{2n}{3}}\right),$$

whereas $p_2(n, \ )$ with $= 0, 1$ (and we expect the same for general) grows half as fast:

$$p_2(n,\ell) \sim \frac{1}{2\sqrt{3}n^{5/4}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Neither grows much faster than ordinary partition numbers [37]:

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(2\pi\sqrt{\frac{n}{6}}\right),$$

since the exponential in both cases is proportional to $\sqrt{n}$.

2.2 Plane partition viewpoint

In the previous section, we saw that the wedge characters of the twisted sector eq. (2.6) can be interpreted as counting configurations of two Young diagrams glued together along their first columns. This viewpoint now allows us to determine the corresponding coset representation using the description of W_∞ representations in terms of plane partitions, which we now review.

Just as partitions of n count the number of ways of drawing Young diagrams with n boxes, plane partitions of n count the number of ways of stacking n boxes in the corner of a room (such that the number of boxes is non-increasing along all three directions, i.e., the projections onto the xy, yz, zx planes all have the shape of a Young diagram). The generating function of plane partitions is the MacMahon function

$$M(q) \equiv \prod_{n=1}^{\infty}(1-q^n)^{-n} = \sum_n M(n)q^n = 1+q+3q^2+6q^3+13q^4+24q^5+48q^6+86q^7+\cdots.$$

From the definition of the MacMahon function, it is immediate that it is identical to the vacuum character of the $W_{1+\infty}$ algebra—for each spin $s = 1, 2, \cdots$, the modes that contribute to the vacuum Verma module are those with $n = -s, -s - 1, \cdots$. It has the asymptotic behavior [38]

$$M(n) \sim \zeta(3)^{7/36} e^{\zeta'(-1)} n^{-25/36} \exp\left(\frac{3\zeta(3)^{1/3}}{2} n^{2/3}\right).$$

Because of the n^{2/3} in the exponent, it grows much faster than the ordinary partition function (2.19) or the two-partition function (2.18), whose exponents are proportional to √n.

More interestingly, we can also consider the set of plane partitions that share a given asymptotic behavior described by (Λ , Λ , Λ_z), where Λ with a = x, y, z is the Young diagram to which the plane partition asymptotes in the limit a → ∞. For a given asymptotic (Λ , Λ , Λ_z), there exists a unique plane partition configuration with the least number of boxes—call it the minimal configuration with this boundary condition. The character N_{Λ ,Λ ,Λ_z}(q) counts the number of ways of stacking boxes starting from this minimal configuration.

When all three asymptotics are trivial, the character N_{plane}(q) reduces to the MacMahon function—the vacuum character of the W_{1+∞} algebra. When at least one of the three Young diagrams is trivial (without loss of generality we may take Λ_z = 0), the generating function of plane partitions reproduces precisely the W_{1+∞} character for the coset representation (Λ , Λ) [18-21]:

$$N^{\text{plane}}_{\Lambda_x,\Lambda_y,0}(q) = \chi^{W_{1+\infty}}_{(\Lambda_x;\Lambda_y)}(q).$$

Here, representations of the coset (2.1) are labeled by pairs $(\Lambda^+;\ \Lambda^-)$, where Λ^+ denotes a representation of su(N) in the numerator, while Λ^- denotes a representation of su(N)_{k+1} in the denominator. (The representation of $\mathrm{su}(N)_1$ is then uniquely fixed by the selection rules.) The interpretation of representations with three non-trivial Young diagrams is not yet entirely clear, although [21] has argued that exchanging the three asymptotic directions reflects precisely the 'triality' symmetry of [22].

We now use the map between the generating function of plane partitions and W_{1+∞} characters (2.22) to identify the coset representations corresponding to twisted sector states. First, the configurations of two glued Young diagrams counted by p_2(n,), see eq. (2.12), can be described in plane partition language as plane partitions with a pit dug at (x, y) = (2, 2) [39]. Here, a 'pit' means one cannot place a box at that position. Since a plane partition must give Young diagrams upon projection along all three directions, a 'pit' at (x, y) = (2, 2) means we cannot place any box at positions with x ≥ 2 or y ≥ 2. Plane partitions with this 'pit' condition therefore reduce to a pair of Young diagrams glued along their first columns, where the two Young diagrams sit in the zx and yz planes, respectively, and the shared first column is along the z-direction.

Recall that eq. (2.12), the plane partition with the 'pit' condition, only counts the wedge character. The full character is obtained by multiplying the wedge character with the vacuum character. Since the first is given by the 'pit' partition function while the second equals the MacMahon function—the plane partition starting from an empty corner—the full coset character is described by the window sill configuration of Figure 2 [Figure 2: see original paper] in the limit where the heights of the walls are taken to infinity. Indeed, in this limit there is a natural separation between configurations involving boxes stacked on the 'floor' —counted by the MacMahon function and describing the contribution of W_{1+∞} modes outside the wedge—and those stacked on the high 'window-sill', counted by the plane partition with a pit at (x, y) = (2, 2).

It remains to relate the height b of the window sill to the twist of the corresponding bosonic representation. By comparing conformal dimensions (see Section 2.4), we find that the relevant coset representations are

$$(\Lambda^+;\Lambda^-) = ([0^{b-1},1,0,\dots,0];[0^{b+\ell-1},1,0,\dots,0])\,,$$

where

$$b = \nu N.$$

Furthermore, the case where more than one boson is twisted is described by putting the relevant window-sills together (see Figure 3 [Figure 3: see original paper] for an example with two twisted bosons). The generalization to situations where some or all bosons are twisted is then straightforward, with the height of each window-sill identified with N, where is the twist parameter of the corresponding boson. In particular, there is a natural separation of different box configurations into boxes on the 'floor' —again describing contributions from outside-the-wedge modes—and boxes stacked on individual window-sills. The wedge character in the multi-twist case is therefore just the product of the individual wedge characters (2.14).

These considerations suggest that the coset representation (Λ^+; Λ^-) corresponding to the ground state of the multi-twisted sector associated with twist ($_1$, ⋯ ,) with $_1$ ≥ $_2$ ≥ ⋯≥ is of the form (Λ^+; Λ^-), where $\Lambda^+ = \Lambda^- = \Lambda$ with c = N boxes in the i-th column. Furthermore, the various twisted excitations (corresponding to non-trivial powers of y ^{ }) are described by coset representations where a finite number of boxes (corresponding to the number of twisted excitation modes) are added to or removed from Λ^- (but not Λ^+). Namely, the column heights of Λ^+ and Λ^- are

$$c_i^+ = \nu_i N, \quad c_i^- = \nu_i N + \ell_i,$$

where is finite (and does not scale with N). These predictions will be tested below following the techniques of [17].

2.3 The null-vector analysis

One key result from the previous two subsections is that the coset representation corresponding to the ground state of the single-twist sector has the form (2.23), given by two totally anti-symmetric Young diagrams both with b boxes, where b $\rightarrow \infty$ in the ‘t Hooft limit. This was based on a character analysis using the description of coset representations in terms of plane partitions. In this subsection, we approach this problem from a ’microscopic’ viewpoint by studying the null-vector structure of the relevant family of representations. As we shall see, this nicely confirms the above results. We concentrate on the ground state representations (2.23) with = 0, since for them the analysis is simplest; the excitation spectrum will be studied in more detail in the following subsection (albeit from a slightly different viewpoint).

Since our considerations are only valid in the ’ t Hooft limit, we can decouple the outside-the-wedge modes and think of these representations as representations of the wedge algebra hs[1]. We therefore seek the hs[1] representation whose character is the wedge character

$$\chi^{[\nu]}_{(\ell)}(q) = q^h \prod_{n=1}^{\infty} (1 - yq^{n-1+\nu})^{-1} (1 - y^{-1} q^{n-\nu})^{-1} \Big|_{y^0}$$

with = 0, which equals

$$\chi^{[\nu]}_{(0)}(q) = q^h \left(1 + q + 3q^2 + 6q^3 + 12q^4 + 21q^5 + 38q^6 + 63q^7 + \cdots\right).$$

Although the wedge character ^{[ ]}_{(0)}(q) was computed using the free boson viewpoint corresponding to hs[1], we shall see that an hs[λ] representation with this character can be constructed for any value of λ, not just $\lambda = 1$. One way to do this is to start with an arbitrary highest weight state ϕ and determine the null-vector structure it must possess to lead to a character of the form (2.27).

Let us denote the modes of the hs[λ] algebra by V^{(s)}_m, where s = 2, 3, ⋯ and |m| $\leq$ s-1. Furthermore, we denote by w^{(s)} the eigenvalue of V^{(s)}_0 acting on a highest weight state ϕ. A generic highest weight representation of hs[λ] can be specified by its charges w^{(s)} for all s $\geq$ 2. However, the representation given by (2.27) is very special: it has only a single descendant at level one—this type of representation was named a ‘level-one representation’ in [12]. For a level-one representation, the condition of having only a single state at level 1 is so strong that it fixes all w^{(s)} with s $\geq$ 4 in terms of its conformal dimension w^{(2)} h and its spin-3 charge w^{(3)}.

To see this, we first note that having only a single state at level 1 means that all V^{(s)}*{-1} ϕ must be proportional to one another, and in particular to V^{(2)}*{-1} ϕ:

$$V_{-1}^{(s)}\phi = s\, w^{(s)} V_{-1}^{(2)}\phi,$$

where the proportionality factor is fixed by requiring that these relations hold upon applying V^{(2)}_1 L_1 to both sides. Then, by taking the commutator with V^{(3)}_1, we can recursively determine the various w^{(s)} eigenvalues in terms of h and w^{(3)}. It is more convenient to use h and the ratio

$$\alpha = \frac{3w^{(3)}}{2h}$$

to express the result; for the first few spins we find explicitly

$$w^{(4)} = \frac{h}{5}\left(\alpha^2 + \alpha\frac{4-\lambda^2}{10-\lambda^2}\right),$$

$$w^{(5)} = \frac{h}{6}\left(\alpha^3 + \alpha\frac{20-3\lambda^2}{10-\lambda^2}\right),$$

$$w^{(6)} = \frac{h}{7}\left(\alpha^4 + \alpha^2\frac{2\lambda^4-20\lambda^2+64}{(10-\lambda^2)^2} + \frac{\lambda^4-20\lambda^2+64}{(10-\lambda^2)^2}\right).$$

These expressions agree with those for the free boson obtained in eqs. (B.5)-(B.7) of [12] upon setting $\lambda = 1$. The level-one condition not only fixes all higher charges in terms of (h, α) but, together with the structure of hs[λ], also imposes very strong constraints on the number of descendants at every level. It was shown recursively in [12] that the wedge character of a generic level-one representation is precisely the MacMahon function (2.20). More specifically, the full representation is generated by modes V^{(s)}*{-n} φ where s = n+1, ⋯, 2n, which matches another form of the MacMahon function, M(q) =* {n=1}∞ (1-q^n)^{-n}.

Comparing the q-expansion of the wedge character (2.27) with the MacMahon function (i.e., the hs[λ] character of a generic level-one representation), we see that the ground state of the twisted sector does not lead to a generic level-one representation: it has a first additional null-vector at level 4—this was already noted in [12]. This property can now be used to determine constraints on the parameters (h, α). We study the structure of null vectors systematically from level 2 up to level 5. Here we give only a brief summary; details are in Appendix A.

At each level, we work out the inner product matrix of the corresponding basis states and determine its determinant. A vanishing determinant signals a null-vector at that level. Any null-vector at level gives rise to descendant null-vectors at higher levels, so we focus on new null-vectors appearing at each level. Up to level 5, these arise for the following (h, α) values:

Level 2: $\alpha = \pm(1 \pm \lambda/2)$, $\alpha = \pm\sqrt{}(8\text{h} + \lambda^2)$

Level 3: $\alpha = \pm(2 \pm \lambda/2)$, $\alpha = \pm\sqrt{}(8\text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(4\text{h} + \lambda^2)$

Level 4: $\alpha = \pm(3 \pm \lambda/2)$, $\alpha = \pm\sqrt{}(8\text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(4\text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(8/3\ \text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(\lambda^2 - 8\text{h})$

Level 5: $\alpha = \pm(4 \pm \lambda/2)$, $\alpha = \pm\sqrt{}(8\text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(4\text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(8/3\ \text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(2\text{h} + \lambda^2)$, $\alpha = \pm\sqrt{}(8/4\ \text{h} + \lambda^2)$

For our case, we are interested in the three new roots (not counting multiplicities and conjugations) at level 4; the ground state of the twisted sector must satisfy at least one of these relations.

All roots except the third pair at level 4 follow a simple pattern, and as explained in Appendix A, these 'standard' roots are attained by finite tensor powers of the minimal representation. It is thus very suggestive that the additional null-vector appearing in the twisted sector representation corresponds to this 'special' root.

To confirm this, we compute (h, α) for the representation

$$(\Lambda^+;\Lambda^-) = ([0^{b-1},1,0,\dots,0];[0^{b-1},1,0,\dots,0])\,,$$

using the Drinfeld-Sokolov approach (see Appendix B for details). From eqs. (B.15) and (B.16), the eigenvalues take the form

$$h = \frac{\lambda^2 b(N+1)(N-b)}{2N^2(N+\lambda)},$$

$$\alpha = \frac{\lambda(N+2)(N-2b)}{2N\sqrt{N(N+\lambda)}}.$$

Taking the ' t Hooft limit and using $0 \leq \lambda \leq 1$ with $0 < \text{b} < \text{N}$ (only Young diagrams of height at most N are allowed), we see that neither of the first two roots can be solved by representations of this type, whereas the last one,

$$\alpha = \pm\sqrt{\lambda^2 - 8h},$$

is solved for any $0 < \text{b} < \text{N}$. Demanding $\text{h} > 0$ in the ' t Hooft limit (i.e., they are not light states) gives

$$b = \nu N, \quad \nu < 1.$$

It remains to understand the meaning of . Returning to the special case $\lambda = 1$ (the free boson) where we started, eq. (2.33) with (2.35) reduces to

$$h = \nu(1-\nu), \quad \alpha = 1 - 2\nu,$$

which agrees with the twisted sector representation of the free boson from [7] (see its eqs. (4.4) and (4.5)). Thus we conclude that   can indeed be identified with the twist parameter.

2.4 The ground state conformal dimension and the excitation spectrum

In this section, we confirm the identification between coset representations and twisted sector states given in eq. (2.25) (for the ground state) and (2.26) (for excited states) by matching their conformal dimensions in the large N limit, following the same strategy as for the N = 2 case in [17].

The conformal dimension of the ground state of the coset representation (Λ^+, Λ^-) is

$$h(\Lambda^+;\Lambda^-) = \frac{C_2(\Lambda^+)}{N+k} + \frac{C_2(\mu)}{N+1} - \frac{C_2(\Lambda^-)}{N+k+1} + n,$$

where C_2 is the quadratic Casimir,   is the $su(N)_1$ weight uniquely determined by the condition that Λ^+ +   - Λ^- lies in the root lattice, and n denotes the first descendant level where Λ^- appears in the affine representation of the numerator.

The Casimir can be written in terms of row lengths r and column heights c as

$$C_2(\Lambda) = \frac{1}{2}\sum_i r_i^2 - \frac{1}{2N}\left(\sum_i r_i\right)^2 + \sum_i r_i(N+1-2i) = \frac{1}{2}\sum_j c_j^2 - \frac{1}{2N}\left(\sum_j c_j\right)^2 + \sum_j c_j(N+1-2j),$$

where B =   r =   c is the total number of boxes.

Let us start with the conformal dimension of the ground state of the twisted sector. Since $\Lambda^+ = \Lambda^- = \Lambda$, the $su(N)_1$ representation   is trivial and n = 0, so the ground state conformal dimension equals

$$h(\Lambda;\Lambda) = \frac{C_2(\Lambda)}{(N+k)(N+k+1)}.$$

We are interested in the large N behavior at fixed k. Since r ≤ k, the r ^{2} term is subleading at large N, and since the total number of boxes scales linearly with N, the same is true for the $B^2/2N$ term. Thus, to leading order in $N \to \infty$,

$$C_2(\Lambda) \sim N\sum_j c_j = N^2 \sum_j \nu_j.$$

Dividing by the denominator in (2.39) then yields

$$h(\Lambda;\Lambda) \sim \sum_j \nu_j(1-\nu_j),$$

which agrees precisely with the usual ground state energy of a multi-twist sector associated with twist ($_1$, ···,   ) with $_1 \geq$ $_2 \geq \cdots \geq$.

Having verified that the large N limit of the coset representation (Λ, Λ) of the form (2.25) agrees with the twisted sector ground state energy, we now compute the excitation spectrum above this ground state. For a generic twisted sector corresponding to $_1 \geq$ $_2 \geq \cdots \geq$, the different bosonic generators have mode numbers -   + m, where j = 1, ···, k and m   . Thus the lowest excitations raise the conformal dimension by   . These excitations should correspond to different ways of adding a single box to Λ^- without modifying Λ^+.

To compute the difference in conformal dimension between the coset representation (Λ; $\Lambda^{( )}$), where $\Lambda^{( )}$ differs from Λ by adding a single box to the i-th column, and the ground state (Λ; Λ), we use eq. (2.37). For (Λ; $\Lambda^{( )}$), the $su(N)_1$ representation   equals the fundamental representation while n remains 0. Thus the conformal dimension difference equals

$$h(\Lambda;\Lambda^{(i)}) - h(\Lambda;\Lambda) = \frac{C_2(\Box)}{N+1} + \frac{C_2(\Lambda^{(i)}) - C_2(\Lambda)}{(N+k)(N+k+1)}.$$

The Casimir of the fundamental is C_2( ) = (N^2-1)/2N, and the difference of the two Casimirs, to leading order in N, can be computed using (2.40):

$$C_2(\Lambda^{(i)}) - C_2(\Lambda) \sim -c_i,$$

where c =   N is the number of boxes in the i-th column of Λ. Hence, in the N → ∞ limit the excitation energy above the ground state takes the form

$$h(\Lambda;\Lambda^{(i)}) - h(\Lambda;\Lambda) \sim \nu_i,$$

as expected. Removing a box from the i-th column of Λ^- changes the sign of the contribution from (2.43), giving excitation energy δh = (1-). This describes the action of the complex conjugate mode. These results reproduce the excitation spectrum in the twisted sector, see particularly eq. (2.10), where   > 0 and   < 0 correspond to excitation by the boson and its complex conjugate, respectively. In plane partition language,   is the difference in height of the window sills of Λ^- relative to Λ^+; thus the action of the boson and its conjugate can be thought of as adding a box to Λ^- and Λ^+, respectively. This identification is only valid in the large N limit where we have a 'Fermi sea' of boxes and the action of anti-boxes can be described as creating a hole. In general, the plane partition viewpoint only describes representations made from boxes; anti-boxes do not appear directly.

3 The twisted sector in the free fermion description

The bosonic coset theories also have a free field description for $\lambda = 0$, where free fermions emerge. Specifically, $\lambda = 0$ corresponds to taking k $\to \infty$ at fixed N (see eq. (2.2)), and the resulting theory can be identified with the u(1) coset of the theory of N complex fermions [33]. These fermions give rise to bilinear currents of the form

$$W^s(z) = n(s)\sum_{l=0}^{s-1}(-1)^l \binom{s-1}{l}^2 \binom{s-1}{s-l} \partial^{s-1-l}\psi_j^\dagger \partial^l \psi^j,$$

where n(s) is an s-dependent normalization constant. The modes of these fields generate the linear W_{1+∞} algebra [29-31].

The bilinear currents are invariant under an SU(N) subgroup, so the resulting theory can be thought of as a continuous orbifold by this group [34]. There are therefore again twisted sectors that should admit a coset description. In the following, we work out the details of this correspondence.

For a single twisted complex fermion with modes

$$\bar{\psi}_{r+\nu}, \quad \psi_{r-\nu}, \quad r \in \mathbb{Z} + \frac{1}{2},$$

the wedge character is given by the fermionic analogue of (2.5):

$$\phi^{[\nu]}(q, y) = q^h \prod_{n=0}^{\infty}(1 + yq^{n-1/2+\nu})(1 + y^{-1}q^{n-1/2-\nu}),$$

where the twist lies in the interval -½ <   < ½. As in the bosonic case, powers of y track the action under the cyclic group, so states with a given power of y furnish a representation of the hs[0] algebra with character

$$\phi^{[\nu]}_{(m)}(q) \equiv \phi^{[\nu]}(q, y)\big|_{y^m}.$$

In contrast to the bosonic case, the fermionic wedge character ϕ_{( )}(q) is much easier to compute. Using the Jacobi triple product identity,

$$\prod_{n=1}^{\infty}(1 - q^n)(1 + yq^{n-1/2})(1 + y^{-1}q^{n-1/2}) = \sum_{n=-\infty}^{\infty} y^n q^{n^2/2},$$

we immediately have

$$\phi^{[\nu]}_{(m)}(q) = q^{h+m\nu+m^2/2}\prod_{n=1}^{\infty}(1-q^n)^{-1} = q^{h+m\nu+m^2/2}\left(1+q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+\cdots\right).$$

In the following, we use the plane partition viewpoint to identify the corresponding coset representations.

3.1 The plane partition viewpoint

Recall from the bosonic analysis of Section 2.2 that from the plane partition perspective, the coset character factorizes into two pieces: one corresponding to stacking boxes on the window sill (accounting precisely for the wedge character) and one corresponding to stacking boxes on the empty floor (giving the MacMahon function that counts outside-the-wedge modes). In the present situation we expect a similar structure, except that now the wedge character ϕ_{(m)}(q) is the generating function of partition numbers—it counts Young diagrams with n boxes. (This is true for all m—the only m-dependence appears in the overall q exponent.)

The most naive guess for the correct plane partition configuration seems to be a single-layer window sill along the x-direction. In this scenario, stacking boxes on this window sill (equivalent to drawing usual two-dimensional Young diagrams) gives the wedge character (3.6), while stacking boxes on the floor gives the MacMahon function, counting outside-the-wedge modes of W_{1+∞}. However, this guess is wrong since the coset representation corresponding to this plane partition, i.e., ([0^{b-1}, 1, 0, ⋯, 0]; 0) with b $\to$ ∞, has conformal dimension (see eq. (B.8))

$$h([0^{b-1},1,0,\ldots,0];0) = \frac{b(N-b)}{2N} + \frac{k+1}{N+k},$$

which diverges in the ’ t Hooft limit.

We can instead reverse the roles of window sill and floor, letting the floor count the wedge character and the window sill count the outside-the-wedge modes. If we take the window sill to be only one box high, it restricts box-stacking on the floor to a counting of Young diagrams. This leads us to plane partitions with a window-sill that is only one box high but has a fat ‘L’ shape whose widths b and b+m are taken large (see Figure 4 [Figure 4: see original paper]). Since the widths are large, boxes placed on top of the window-sill near the origin are again counted by the MacMahon function (describing outside-the-wedge modes of W_{1+∞}). On the other hand, configurations involving only boxes on the floor are counted by plane partitions satisfying a (generalized) ‘pit’ condition, where the pit is now located at (x, y, z) = (1, 1, 2) and prevents any boxes from being stacked vertically.

The coset representations corresponding to this type of plane partition asymptotics are

$$([b, 0, \dots, 0]; [b + m, 0, \dots, 0])$$

for sufficiently large b. The neutral sector, whose wedge character is ϕ^{[ ]}_{(0)}(q), corresponds to m = 0 (see Figure 4(a)); higher representations with m > 0 are shown in Figure 4(b).

In all these cases, the relevant representations are level-one representations with one null-vector at level 2 (i.e., only 2 rather than 3 states at the second descendant level). All these representations have α = √(8h + λ^2) in the ’ t Hooft limit, as follows from the analysis of Appendix B (see e.g., eq. (B.20)). It is also not difficult to guess which plane partition asymptotics describe the case where more than one fermion is twisted: as in the bosonic case, we simply put different such diagrams together (see Figure 5 [Figure 5: see original paper] for the case with two twisted fermions).

The coset representation describing the ground state of a generic twist is then of the form

$$([b_1, b_2, b_3, \dots, b_{N-1}]; [b_1, b_2, b_3, \dots, b_{N-1}]) \,,$$

where the associated twists are

$$\nu_i = \frac{b_i - b_{i-1}}{k},$$

and r is the number of boxes in the i-th row (i = 1, ⋯, N), while B =   r is the total number of boxes. This identification implies that the sum of all N twists is zero, as must be the case for a group element in SU(N). By applying a field identification transformation if necessary, we may assume all r ≤ k/2, thus implying -½ <   ≤ ½.

3.2 The ground state conformal dimension and the excitation spectrum

As in the bosonic case, we can confirm these claims by direct CFT calculations. We first show that the coset representation (3.9) has the correct conformal dimension

$$h = \sum_i \frac{\nu_i^2}{2}$$

in the k → ∞ limit. Note that there are N complex fermions, and each -twisted fermion contributes ^{2}/2 to the ground state conformal dimension. In relating the free fermion theory to the coset at $\lambda = 0$, we need to take the u(1) orbifold, i.e., subtract -u^2/2N from the conformal dimension of the numerator, where u is the u(1)-charge of the state (see eq. (3.15) below). Since = 0, the u(1)-charge is u = 0, giving (3.11).

To derive this from the coset viewpoint, note that the coset representation of the ground state has the form (Λ; Λ), where the row lengths r of Λ satisfy (3.10). In the orthogonal basis of Appendix B (see particularly eq. (B.1)), the quadratic Casimir of Λ is

$$C(\Lambda) = \sum_i \left(r_i - \frac{B}{N}\right)^2 + \sum_i r_i(N+1-2i) \sim \sum_i r_i^2 - \frac{B^2}{N}.$$

Alternatively, this follows from (2.38) since the r ^{2} and -B^2/2N terms are the only expressions proportional to k^2. Dividing by (N+k)(N+k+1) k^2 in the large k limit yields the desired conformal dimension (3.11).

Next, we study the excitation spectrum following the same logic as for the bosonic case (eq. (2.42)). From the difference of Casimirs we now find, to leading order in k,

$$C_2(\Lambda^{(i)}) - C_2(\Lambda) \sim ((r_i+1)^2 - r_i^2) - \frac{(B+1)^2 - B^2}{N} \sim r_i - \frac{B}{N} = -k\nu_i,$$

where Λ$^{( )}$ is the representation with an additional box in the i-th row. Thus the excitation spectrum is

$$h(\Lambda;\Lambda^{(i)}) - h(\Lambda;\Lambda) \sim \nu_i + \frac{N-1}{2N}.$$

For the free fermion theory one would expect + ½; the discrepancy arises because to obtain the W∞[0] theory described by the coset in the k → ∞ limit, one must divide out a u(1) algebra—this was explained in detail in [33]. In particular, the decoupled stress-energy tensor is

$$\tilde{T} = T - \frac{1}{N} : JJ :,$$

and since individual fermions carry unit u(1) charge, they define primary fields of conformal dimension

$$\Delta = \frac{1}{2} + \frac{N-1}{2N}$$

in the decoupled theory. (Another way to reach this conclusion is by observing that the conformal dimension of the (0;) representation equals (N-1)/2N in the large k limit.) This accounts precisely for the additional term in (3.14). Removing a box from the i-th row changes the sign in (3.13), leading to - + (N-1)/2N instead of (3.14), which describes the action of the conjugate fermionic mode (see eq. (3.3)).

4 Conclusions

In this paper, we have identified the twisted sector states of bosonic higher spin CFTs in terms of coset representations. The main idea was to use the description of W∞ representations in terms of plane partitions [18–21]. Our analysis demonstrates that this method provides a powerful approach for characterizing these representations, and this perspective is likely to have other important applications.

The crucial plane partition configurations allow a natural separation into contributions from wedge modes and those from outside-the-wedge modes (see the discussion in Section 2.2). In particular, the former has a nice combinatorial description in terms of plane partitions with (generalized) pit conditions. It would be interesting to see whether this observation generalizes and whether the representation theory of the wedge algebra hs[λ] can generally be captured by plane partitions with suitable 'pit' conditions.

Currently, the plane partition viewpoint has only been developed for the W∞ algebra appearing in the duality to bosonic higher spin theory; it would be very interesting to generalize this technique to supersymmetric cases. In particular, the N = 2 case where Young super tableaux [45, 46] naturally appear (see e.g., [47]) should allow a nice generalization.

Since plane partitions also describe the representation theory of the affine Yangian algebra of gl_1, which is believed to contain W∞ as a subalgebra [21], this viewpoint relates higher spin symmetries to Yangian symmetries that typically arise in integrable systems. This approach may therefore pave the way toward understanding the relation between higher spin symmetries and integrability. As with the embedding into string theory, the sharpest results will likely be possible in the maximally supersymmetric N = 4 case, so finding the appropriate N = 4 supersymmetric generalization of the affine Yangian would be very interesting.

The coset representations we found exist for generic N and k, while a direct twisted sector interpretation is only possible for the free field cases $\lambda = 0$ (free fermions) and $\lambda = 1$ (free bosons). It would be interesting to understand whether these representations also have a natural interpretation away from these points, for example in terms of parafermions. Finally, it is intriguing that the structure of the coset representations is very similar in both cases to those appearing in the extension from the higher spin algebra to the Higher Spin Square (see eqs.

(3.1) and (3.10) in [11]). Understanding the reason underlying this similarity would be very worthwhile.

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A The general form of level-1 representations

In this appendix, we study the structure of a generic level-1 representation at arbitrary λ through level-by-level analysis of their Kac determinants. Here V^{(s)}_m denotes the hs[λ] generator of spin s and mode m.

Level 2: A general level-1 representation has three hs[λ] descendants: V^{(3)}{-2}ϕ, V{(4)}{-2}ϕ, and V^{(2)}{-1}V{(2)}{-1}ϕ. (It is not difficult to show that any other hs[λ] descendant can be written as a linear combination of these using the hs[λ] algebra commutation relations and the fact that there is only a single state at level 1.) The inner product matrix of these states has determinant

$$\det(M_2) = 16h^3 \left(\alpha - 1 - \frac{\lambda}{2}\right) \left(\alpha - 1 + \frac{\lambda}{2}\right) \left(\alpha + 1 - \frac{\lambda}{2}\right) \left(\alpha + 1 + \frac{\lambda}{2}\right) \left(\alpha^2 - 2h - \frac{\lambda^2}{4}\right).$$

Thus zeros appear at $\alpha = \pm(1 \pm \lambda/2)$ and $\alpha = \pm$√(8h + λ^2), each with single multiplicity. (For the first expression, the two minus signs are uncorrelated, describing four different roots.) The overall sign of α relates conjugate representations (since V^{(3)}_0 eigenvalues have opposite signs for conjugate representations while conformal dimension remains the same).

These roots have a simple interpretation in terms of familiar coset representations. From the analysis of [40], the minimal representations (; 0) and (0;) (and their conjugates) have eigenvalues

$$h(\Box; 0) = \frac{1}{2}(1 + \lambda), \quad w^{(3)}(\Box; 0) = -\frac{1}{3}(1 + \lambda)(2 + \lambda),$$

$$h(0; \Box) = \frac{1}{2}(1 - \lambda), \quad w^{(3)}(0; \Box) = \frac{1}{3}(1 - \lambda)(2 - \lambda).$$

The corresponding α values are α(; 0) = -√(8h + λ^2) and α(0;) = √(8h + λ^2). These account for the first four zeros of (A.3), including conjugate representations. They also solve the last two equations of (A.3) since for h =

½$(1 \pm \lambda)$, we have $4 \pm 4\lambda + \lambda^2 = 8\text{h} + \lambda^2 = (2 \pm \lambda)^2$. This reflects that minimal representations (; 0) and (0;) (or their conjugates) have two null-vectors at level 2—they have only a single wedge descendant at this level.

To identify representations with only a single null-vector (corresponding to a single zero) at level 2, note that symmetric tensor powers of minimal hs[λ] representations satisfy

$$h([0^{m-1}, 1, 0, \dots, 0]; 0) = \frac{m}{2}(1{+}\lambda), \quad w^{(3)}([0^{m-1}, 1, 0, \dots, 0]; 0) = -\frac{m}{3}(1{+}\lambda)(2{+}\lambda),$$

$$h(0; [0^{m-1}, 1, 0, \dots, 0]) = \frac{m}{2}(1{-}\lambda), \quad w^{(3)}(0; [0^{m-1}, 1, 0, \dots, 0]) = \frac{m}{3}(1{-}\lambda)(2{-}\lambda).$$

These account for one of the first four zeros of (A.3), including conjugates. However, for m $\geq$ 2 they no longer satisfy the last two zeros, compatible with the fact that the corresponding wedge representations have two states at level 2 (only a single null-vector).

A representation satisfying only one of the last two zeros is given by the two-fold anti-symmetric tensor power of the minimal representation. From Appendix B.3 of [41],

$$h([2, 0, \dots, 0]; 0) = 1 + \frac{\lambda}{2}, \quad w^{(3)}([2, 0, \dots, 0]; 0) = -\frac{1}{3}(2 + \lambda)(4 + \lambda),$$

so α takes the value α([0,1,0,⋯,0];0) = -√(($4{+}\lambda)^2$) = -√(8($2{+}\lambda$) + λ^2).

Level 3: We have similarly determined the Kac determinant at higher levels. At level 3, there are generically 6 independent vectors with basis

$$V^{(4)}_{-3}\phi, \quad V^{(5)}_{-3}\phi, \quad V^{(6)}_{-3}\phi, \quad V^{(2)}_{-1}V^{(3)}_{-2}\phi, \quad V^{(2)}_{-1}V^{(4)}_{-2}\phi, \quad V^{(2)}_{-1}V^{(2)}_{-1}V^{(2)}_{-1}\phi.$$

The Kac determinant is

$$\det(M_3) = (2\alpha{-}\lambda{-}4)(2\alpha{-}\lambda{-}2)^3(2\alpha{-}\lambda{+}2)^3(2\alpha{-}\lambda{+}4)(2\alpha{+}\lambda{-}4)(2\alpha{+}\lambda{-}2)^3(2\alpha{+}\lambda{+}2)^3(2\alpha{+}\lambda{+}4)(-4\alpha^2{+}4h$$

with 24 zeros at $\alpha = \pm(2 \pm \lambda/2)$, $\alpha = \pm$√(8h + λ^2), and $\alpha = \pm$√(4h + λ^2). Zeros appearing already at level 2 (the first line) have multiplicity 3, while new zeros have multiplicity one. The new roots are satisfied for the anti-symmetric two-fold tensor product of the minimal representation (see eq. (A.11))—together with the representation associated to the other minimal representation and their conjugates, this accounts for the first four new roots. The last two new roots are attained for the 2-fold symmetric tensor power of the minimal representation (see eqs. (A.8) and (A.9) with m = 2).

Level 4: At level 4, there are generically 13 states, and the 78 roots of the corresponding Kac determinant (including multiplicities) are

$$\alpha = \pm\left(3 \pm \frac{\lambda}{2}\right), \quad \alpha = \pm\sqrt{8h+\lambda^2}, \quad \alpha = \pm\sqrt{4h+\lambda^2}, \quad \alpha = \pm\sqrt{\frac{8}{3}h+\lambda^2}, \quad \alpha = \pm\sqrt{\lambda^2-8h}.$$

The roots in (A.15) appear already at level 2, while those in (A.15) and (A.16) appear at level 3; these therefore have higher multiplicity. The new roots (appearing with multiplicity one) are associated with solutions (A.17) and (A.18). The roots in (A.17) correspond to the totally anti-symmetric three-fold tensor product of the minimal representation (accounting for the first four roots) and the totally symmetric three-fold tensor product (accounting for the last two roots). The roots in (A.18) are of a different form and are relevant for the bosonic twisted representation (see eqs. (B.15)-(B.16)).

Level 5: We have also performed the analysis at level 5, where the generic level-one representation has 24 states and the Kac determinant has 192 roots (including multiplicities). In addition to roots appearing already at level 4 (see eqs. (A.15)-(A.18)), the new roots at level 5 are of the form

$$\alpha = \pm\sqrt{\frac{8}{4}h+\lambda^2},$$

and thus have the same structural form as in (A.15)-(A.17). They correspond to the totally symmetric and anti-symmetric four-fold tensor products of the minimal representation.

B The spin 3 charge of some simple representations

To identify representations realizing these roots, we need to calculate both conformal dimension and w^{(3)} eigenvalue for level-one representations. In this section we recall how the spin-3 charge can be determined using the Drinfeld-Sokolov approach.

Coset representations are labeled by pairs of su(N) representations (Λ^+; Λ^-). For each such Λ, denote by r the number of boxes in the i-th row of the corresponding Young diagram. Then in orthonormal basis, the weight Λ has components (see e.g., Appendix A of [41])

$$\Lambda_i = r_i - \frac{B}{N},$$

where B = r is the total number of boxes (and r_N 0). The Weyl vector has components

$$\rho_i = \frac{N+1}{2} - i.$$

By construction, Λ = 0 and   = 0. Following [42], we define the vector as

$$\theta = \alpha_+(\Lambda^+ + \rho) + \alpha_-(\Lambda^- + \rho),$$

where

$$\alpha_+ = \sqrt{\frac{N+k+1}{N+k}}, \quad \alpha_- = -\sqrt{\frac{N+k}{N+k+1}}.$$

The power sums C ( ) are

$$C_s(\theta) = \sum_i (\theta_i)^s = (-1)^{s-1} s \sum_{i_1 < i_2 < \cdots < i_s} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_s}.$$

In terms of these, the conformal dimension and spin-3 charge of $(\Lambda^+; \Lambda^-)$ in the 'primary basis' are

$$h(\Lambda^+; \Lambda^-) = C_2(\theta) + \frac{N(N^2-1)}{24(N+k)(N+k+1)},$$

$$w^{(3)}(\Lambda^+; \Lambda^-) = C_3(\theta).$$

As a consistency check, for the minimal representation $\Lambda^+ =$, we find

$$h(\Box; 0) = \frac{N-1}{2(N+k)} + \frac{N(N^2-1)}{24(N+k)(N+k+1)},$$

$$w^{(3)}(\Box; 0) = \frac{(N-1)(N-2)}{3(N+k)^{3/2}(N+k+1)^{1/2}}.$$

Similarly, for the representation ([0^{b-1}, 1, 0, ···, 0]; 0),

$$h([0^{b-1}, 1, 0, \dots, 0]; 0) = \frac{b(N-b)}{N+k} + \frac{N(N^2-1)}{24(N+k)(N+k+1)},$$

$$w^{(3)}([0^{b-1}, 1, 0, \dots, 0]; 0) = \frac{b(N-b)(N-2b)}{(N+k)^{3/2}(N+k+1)^{1/2}}.$$

In the ’ t Hooft limit (for finite b   N, k),

$$h([0^{b-1},1,0,\ldots,0];0) \sim b\,h(\square;0) \sim \frac{b}{2}(1+\lambda),$$

$$w^{(3)}([0^{b-1},1,0,\ldots,0];0) \sim b\,w^{(3)}(\square;0) \sim \frac{b}{3}(1+\lambda)(2+\lambda).$$

This approach also allows calculation of charges for representations where both Λ^+ and Λ^- are non-trivial. For example, for the representation (;),

$$h(\square;\square) = \frac{(N-1)(N+1)}{2N(N+k)(N+k+1)} + \frac{N(N^2-1)}{24(N+k)(N+k+1)},$$

$$w^{(3)}(\square;\square) = \frac{(N-2)(N-1)(N+1)(N+2)}{6N^2(N+k)^{3/2}(N+k+1)^{3/2}}.$$

In the ’ t Hooft limit,

$$\alpha(\square;\square) \sim \frac{3w^{(3)}}{2h} \sim \frac{(N-2)(N+2)}{N(N+k)} \sim \frac{1-\lambda^2}{\lambda}.$$

Note that the correct α value in the ’ t Hooft limit can only be determined from the exact (N, k) expressions since both h and w^{(3)} separately vanish in the limit.

For representations describing ground states of a single twisted boson, i.e., ([0^{b-1}, 1, 0, ···, 0]; [0^{b-1}, 1, 0, ···, 0]), we find

$$h = \frac{b(N+1)(N-b)}{2N(N+k)(N+k+1)} + \frac{N(N^2-1)}{24(N+k)(N+k+1)},$$

$$\alpha = \frac{\lambda(N+2)(N-2b)}{2N\sqrt{N(N+\lambda)}}.$$

As explained in eq. (2.33), in the ’ t Hooft limit this solves the root of eq. (A.18) that appears first at level 4.

For representations describing ground states of a single twisted fermion, i.e., ([b, 0, ···, 0]; [b, 0, ···, 0]), we find instead

$$h = \frac{b(N-1)(N+b)}{2N(N+k)(N+k+1)} + \frac{N(N^2-1)}{24(N+k)(N+k+1)},$$

$$\alpha = \frac{\lambda(N-2)(N+2b)}{2N\sqrt{N(N+\lambda)}}.$$

Setting b = k gives, in the ’ t Hooft limit,

$$h(b=\nu k) \sim \frac{\nu^2}{2}, \quad \alpha(b=\nu k) \sim \sqrt{\lambda^2 + 8h(b=\nu k)},$$

which is a root first appearing at level 2 (see eq. (A.3)).

C Combinatorial description of wedge characters

In this appendix, we outline a proof of the combinatorial identity between p_2(n,) and p_2(n) in eq. (2.15). The = 0 case is given and proven in [35]; we generalize it to generic . First consider ≥ 0.

Recall that p_2(n,) counts configurations obtained from a pair of Young diagrams {Γ^+; Γ^-}_{n, } with first columns of height $c^+{}_1$ = $c^-{}_1$ + by gluing them along their common first column and removing the superfluous boxes of Γ^+ from the bottom (see Figure 1). Here n is the number of ‘visible’ boxes after gluing.

On the other hand, p_2(n) counts ordered pairs of Young diagrams ($\Gamma^{(1)}$, $\Gamma^{(2)}$)*n whose total number of boxes is n. To each such pair we can associate an element {Γ^+; Γ^-}*{n, } by shifting $\Gamma^{(2)}$ steps upward and placing it to the right of $\Gamma^{(1)}$ without column overlap. As long as $c^{(1)}{}_1$ + ≤ $c^{(2)}{}_1$, we choose the first column of $\Gamma^{(2)}$ as the ‘shared’ column of {Γ^+; Γ^-}_{n, }; otherwise we move boxes from the bottom of the first column of $\Gamma^{(1)}$ to the top so this column becomes the ‘shared’ column.

This map is well-defined and surjective but not injective. In particular, if $c^+{}_1$ > $c^-{}_2$ + , there are precisely two pairs of Young diagrams ($\Gamma^{(1)}$, $\Gamma^{(2)}$)*n giving the same configuration {Γ^+; Γ^-}*{n, }—we can move boxes from the top of the first column of $\Gamma^{(2)}$ to the bottom, then adjoin the corresponding column to $\Gamma^{(1)}$. Therefore, starting from p_2(n), we must subtract p_2(n- -1) since overcounted diagrams can all be constructed from configurations ($\Gamma^{(1)}$, $\Gamma^{(2)}$)*{n- -1}: we simply add +1 boxes to the first column of $\Gamma^{(2)}$, guaranteeing $c^-{}_1$ > $c^-{}_2$ + in the resulting {Γ^+; Γ^-}*{n, } configuration. But subtracting p_2(n- -1) is excessive—we must add back configurations where $c^-{}_2$ > $c^-{}_3$ + , captured by ($\Gamma^{(1)}$, $\Gamma^{(2)}$)_{n-(+1)-(+2)} since we must put +1 boxes on the second column of $\Gamma^{(2)}$ and +2 boxes on the first to end up with $c^-{}_1$ > $c^-{}_2$ > $c^-{}_3$ + . Recursively proceeding, we arrive at

$$p_2(n,\ell) = \sum_{m=0}^{\infty}(-1)^m p_2\left(n - \sum_{k=\ell+1}^{\ell+m} k\right), \quad \ell \geq 0.$$

This proves eq. (2.15) for ≥ 0; the < 0 argument is identical upon interchanging Γ^+ Γ^- and $\Gamma^{(1)}$ $\Gamma^{(2)}$.

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