Linear Interpolation of Shape Operators for Umbilical Points via Local Parameterization

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Full Text

Preamble

Linear Interpolation of Shape Operators for Umbilical Points via Local Parameterization

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Abstract

Principal curvatures and principal directions are exploited ubiquitously in shape analysis. On one hand, umbilical points severely hinder the analysis as singularities in the vector field of principal directions; on the other hand, providing shape-intrinsic qualitative information about a surface, they are desirable quantities in some applications. Umbilical points are fundamental for geometric analysis, but their accurate computation on a discrete surface remains challenging. In this paper, we develop a simple and effective method to detect umbilical points on a triangle mesh, for which any parameterization works. In particular, we propose two practical processes for local parameterization by orthogonally projecting or conformally transforming the matrix onto a specified parametric plane. Furthermore, we conduct a systematic analysis of our method and demonstrate its convergence behavior. The algorithm is flexible and easy to implement for triangular meshes of arbitrary topology.

Key words: Linear interpolation, Local parameterization, Principal curvature, Principal direction, Shape operator, Umbilical point

1. Introduction

Principal curvatures and their corresponding directions are basic entities for shape analysis because they entirely exhibit the variation of curvature across a smooth two-manifold surface in three-dimensional Euclidean space. Their calculation is a fundamental issue for numerous geometric applications including shape retrieval, shape registration, surface remeshing, surface segmentation, and shape design.

An umbilical point is a singularity with anomalous behavior in the vector field of principal directions. It is of theoretical interest because lines of curvature, which integrate the field, become complicated near it. In applications favoring the orthogonality of the field, such as feature-aligned remeshing [**?**, **?**], umbilical points cause difficulties; we must isolate them by design to deal with them separately. Umbilical points are invariant features of the surface’ s differential structure, acting like fingerprints across the surface [**?**]; hence, their reliable determination is also significant for quantitative analysis in other settings, such as describing or identifying objects [**?**], shape interrogation [**?**], ship-hull design, and progressive additional lens design [**?**]. Although a wealth of estimators for differential quantities have been proposed in the vast literature of discrete geometry, the literature lacks a critical examination regarding how to detect umbilical points over a mesh as opposed to a smooth surface.

The purpose of this work is to investigate a mathematically sound and computationally efficient approach to umbilical point detection over a mesh model of arbitrary topology. The remainder of this paper is organized as follows. In the next section, we provide a summary of related work and their mathematical background on the numerical computation of umbilical points over a mesh. In Section 3, we describe mathematical preliminaries in classical differential geometry concerning the characteristics of umbilical points and shape operators, and then develop the theoretical foundation of our method to linearly interpolate shape operators for umbilical points. Section 4 proposes algorithms to re-formulate shape operators through a local transformation onto a parameter plane to facilitate linear interpolation. Section 5 presents a unified error analysis of our method. The final two sections present experimental results with discussion and conclude the paper, respectively.

2. Previous Work

Umbilical points have been studied intensively. We shall restate those works from the viewpoint of numerical computation. Interested readers can refer to [**?**, **?**, **?**, **?**] and references therein for more details.

In the following text, a bold uppercase letter represents a matrix and a bold lowercase letter represents a column vector.

2.1. An Umbilical Point

Given a unit normal vector at a point p on a smooth surface, a normal plane at p is one that contains the normal and therefore also contains a unique direction tangent to the surface, cutting the surface in a plane curve. This curve usually has different curvature for different normal planes at p. The two principal curvatures at p, denoted as κ_1 and κ_2, are the maximal and minimal values, respectively. The pioneering Swiss mathematician Leonhard Euler first discovered that the two principal directions of $\mathbf{d}_1$ and $\mathbf{d}_2$ at p, corresponding to the tangent directions realizing κ_1 and κ_2, respectively, are mutually perpendicular. The normal curvature κ_n in the tangent direction making an angle θ with $\mathbf{d}_1$ is characterized by the well-known Euler' s formula: $\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$. If $\kappa_1 = \kappa_2$, however, κ_n in all directions is the same; this point is called an umbilical point and each direction is principal. Umbilical points exhibit special properties and their computation is an important topic in shape analysis. Two commonly used approaches are related to two properties of umbilical points, respectively:

1. In classical differential geometry, an umbilical point is defined as a point where κ_1 and κ_2 are equal. A trivial algorithm to locate an umbilical point is to locally minimize $f = |\kappa_1 - \kappa_2|$ at a vertex against other vertices adjacent to it, or to have the ratio of κ_1 and κ_2 larger than a threshold [**?**]. Its equivalent form of $g = \kappa_M^2 - \kappa_G$ is more frequently exploited in practice, where $\kappa_M = (\kappa_1 + \kappa_2)/2$ is the mean curvature and $\kappa_G = \kappa_1\kappa_2$ the Gaussian curvature. Because the notion of continuous geometric attributes on a smooth surface does not match those on its piecewise linear counterpart, it is uncommon to determine an umbilical point by f or g alone on a sampled surface. This method is more widely utilized for an analytical surface on which numerical optimization can be performed to situate an umbilical point accurately. For instance, a method was proposed in [**?**, **?**] for a free-form surface by solving a system of polynomial equations. For an implicit surface, novel evaluation functions and numerical algorithms have been developed to extract umbilical points in [**?**].

2. Another common method is based on the index of principal direction field at a specified point p. The index of an umbilical point describes the way the lines of curvature turn around the umbilical point and can be computed by: $\frac{1}{2\pi}\oint_c \nabla\theta(c)\cdot dc$ where θ is the angle through which the coordinate axes would have to be rotated to align them with the local principal axes at p [?]. The integral is taken over a small counterclockwise circuit c around p. In implementation, the computation is simplified by pulling the field back onto the tangent plane at p and computing the circuit there [?, ?]: $\frac{1}{2\pi}\oint_c d\theta$ where θ is the angle between a direction of the field and some fixed direction in the parametric plane. The integral is now taken going around the circuit c around p but in the plane. The index of a generic umbilical point is either $-\frac{1}{2}$ or $+\frac{1}{2}$. Thus this method provides a conclusive test for an umbilical point. If the index is zero, no umbilical point is present.

2.2. Linear Interpolation for an Umbilical Point

Localizing an umbilical point is more than just a matter of evaluating $|\kappa_1 - \kappa_2|$ or calculating the index. Because it is likely that the actual umbilical point will fall between the sampling points, the structure of the principal direction field should be taken into account [?]. A method involving linear interpolation of some quantities at vertices of a geometric primitive–for example, a triangle over a mesh–is sought to yield an umbilical point.

In a 2D symmetric tensor field $\mathbf{T}$ of type 2×2, the location of a singular point on a discrete grid is reduced to an interpolating point of zero using bilinear interpolation of the tensor-matrix components between the grid vertices [?]; later generalized to linear interpolation within each triangle $\triangle abc$ by solving a linear system of $\mathbf{T}$ [?]:

$$x\mathbf{D}_a + y\mathbf{D}_b + (1 - x - y)\mathbf{D}_c = \mathbf{0}$$

where the deviator part $\mathbf{D}$ of $\mathbf{T}$ is defined as $\mathbf{D} = \mathbf{T} - \frac{1}{2}\mathrm{tr}(\mathbf{T})\mathbf{E}$; here $\mathbf{E}$ is an identity matrix of 2×2. This method is then employed in remeshing for an umbilical point over a triangular mesh by flattening it to a 2D parametric plane and by interpolating the curvature tensor [?]. For a parametric surface, linear interpolation of curvature tensors in the parametric domain is advantageous for fixing an umbilical point inside a primitive cell rather than only on a vertex as the former two methods do. Their practice, however, has to rely on a globally conformal parameterization of the mesh in order to obtain symmetrical curvature tensors in the parametric domain. Parameterization is a central issue in computer graphics and it is a nontrivial task to obtain a conformal one. Furthermore, a parametrization is not indispensable in most applications. Needless to say, the global parametrization requires the mesh surface to be equal to a planar disk or a planar square in topology; additional errors will be produced in the parametrization, too.

How to estimate an umbilical point with a local parameterization is the key to popularizing the method through linearly interpolating certain quantities at vertices of a geometric primitive. The contributions of this work constitute a novel method of local computation for umbilical points on a triangular mesh. It significantly frees the interpolation from both symmetrical tensors and conformal parameterization so that the underlying surface can be of arbitrary topology. We also carry out a mathematical analysis examining its numerical and convergence behaviors for estimating an umbilical point on a discrete mesh surface—a method that has not yet appeared in the extant literature.

3. Mathematical Preliminaries

Although our method can be portrayed compactly, its mathematics is not described explicitly in the literature. To provide a practical foundation, a brief review of important mathematical preliminaries related to the algorithm is in order.

3.1. Background of Differential Geometry

We summarize the relevant definitions and results about lines of curvature on a parametric surface to describe our method. Differential geometry in parametric representation has been developed richly, and most facts can be found in a standard textbook on classical differential geometry when proofs are not given explicitly.

Given a variable vector $\mathbf{u} = [u; v]^T$, in which u and v are local curvilinear coordinates, a vector-valued parameterized surface in terms of $\mathbf{u}$ is represented as $\mathbf{r}(\mathbf{u}) = [x(u; v); y(u; v); z(u; v)]^T$. If $\mathbf{r}$ is regular, $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent and form a basis of the tangent plane at $\mathbf{r}(\mathbf{u})$. The shape operator $\mathbf{S}$ for $\mathbf{r}(\mathbf{u})$ is $\mathbf{S} = \mathbf{I}^{-1}\mathbf{II}$, where $\mathbf{I} = \begin{bmatrix} \mathbf{r}_u \cdot \mathbf{r}_u & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_u \cdot \mathbf{r}_v & \mathbf{r}_v \cdot \mathbf{r}_v \end{bmatrix}$ and $\mathbf{II} = \begin{bmatrix} \mathbf{n} \cdot \mathbf{r}_{uu} & \mathbf{n} \cdot \mathbf{r}_{uv} \\ \mathbf{n} \cdot \mathbf{r}_{uv} & \mathbf{n} \cdot \mathbf{r}_{vv} \end{bmatrix}$ are the matrices of the first and second fundamental forms in the basis $\mathbf{r}_u$ and $\mathbf{r}_v$, respectively; $\mathbf{n}$ is the unit normal vector.

A basic property of the shape operator is $\mathbf{S}\mathbf{d}_i = \kappa_i \mathbf{d}_i$, $i = 1, 2$, where κ_i and $\mathbf{d}_i$ are the principal curvatures and the corresponding principal directions.

The differential equation of lines of curvature in coordinates is given by $Udu^2 + Vdudv + Wdv^2 = 0$ where $U = EM - FL$, $V = EN - GL$, and $W = FN - GM$.

Therefore, we have a well-known result of an umbilical point in classical differential geometry:

Lemma 3.1. $\mathbf{r}(\mathbf{u})$ is an umbilical point if and only if $U = V = W = 0$ at $\mathbf{u}$.

It seems that $U = V = W = 0$ are over-determined to situate an umbilical point; nevertheless, they are not independent, as shown by the following corollary:

Corollary 3.2. $\mathbf{r}(\mathbf{u})$ is an umbilical point if and only if $V = 0$ at $\mathbf{u}$.

Proof. $V = 0 \Leftrightarrow \frac{EM-FL}{EN-GL} = \frac{FN-GM}{EM-FL} \Leftrightarrow \frac{L}{E} = \frac{M}{F} = \frac{N}{G}$. We verify this lemma immediately following from Corollary 3.2.

Remark 3.3. In the sketch proof of Corollary 3.2, we use a concentrated notion of $\frac{L}{G}$ for the sake of a compact proof. It infers a presupposition that M must be zero if F is zero for this equation to hold. A reader can examine that it does not affect the whole validity of the proof. We should also note that both E and G are always positive numbers for a regular $\mathbf{r}$.

Corollary 3.2 is inconvenient to characterize an umbilical point in numerical computation for a discrete mesh model. Instead, defining $\mathbf{t} = [S_{11} - S_{22}; S_{12} + S_{21}]^T$, a more intuitive and practical condition can be derived:

Lemma 3.4. p is an umbilical point if and only if $\mathbf{t} = \mathbf{0}$.

Proof. $\mathbf{t} = \mathbf{0} \Leftrightarrow S_{11} - S_{22} = 0 \wedge S_{12} + S_{21} = 0 \Leftrightarrow V = 0$. We verify this lemma immediately following from Corollary 3.2.

To the best of our knowledge, Lemma 3.4 is not described explicitly in the literature. Its power will be reflected by our new algorithm to compute umbilical points over a two-manifold mesh model with arbitrary topology.

3.2. Linear Interpolation of Shape Operators for Umbilical Points

In this subsection, we further establish an essential link between a shape operator and a curvature tensor under a conformal parameterization. We then develop a novel approach to linear interpolation based on shape operators for umbilical points rather than curvature tensors, which can relax the restriction of symmetry required for curvature tensors.

3.2.1. A Shape Operator and a Curvature Tensor In contrast to a shape operator, κ_i and $\mathbf{d}_i$ $(i = 1, 2)$ together are also thought of as the curvature tensor $\mathbf{C} = \kappa_1 \mathbf{d}_1 \mathbf{d}_1^T + \kappa_2 \mathbf{d}_2 \mathbf{d}_2^T$ in the computer graphics community. The tensor field $\mathbf{T}$ in Eq.(3) is set as $\mathbf{C}$ for umbilical points, but $\mathbf{C}$ must be symmetrical, which is too strict to exploit in applications. Because both $\mathbf{S}$ and $\mathbf{C}$ are closely related to κ_i and $\mathbf{d}_i$, $i = 1, 2$, there must be a connection between them, and a well-founded thought is to replace $\mathbf{C}$ with $\mathbf{S}$ to take full advantage of shape operators, which have been well studied in differential geometry.

The following proposition reveals the relationship between $\mathbf{S}$ and $\mathbf{C}$:

Proposition 3.5. $\mathbf{C} = \mathbf{S}$ if both are represented in conformal curvilinear coordinates.

Proof. Let $(u; v)$ be conformal coordinates. $\mathbf{d}_1$ and $\mathbf{d}_2$ are unit vectors of principal directions in coordinates. Because $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$, $\mathbf{d}_1 \mathbf{d}_1^T + \mathbf{d}_2 \mathbf{d}_2^T = \mathbf{E}$ holds. Thus $\mathbf{C} = \kappa_1 \mathbf{d}_1 \mathbf{d}_1^T + \kappa_2 \mathbf{d}_2 \mathbf{d}_2^T = \kappa_1 \mathbf{d}_1 \mathbf{d}_1^T + \kappa_2 (\mathbf{E} - \mathbf{d}_1 \mathbf{d}_1^T) = \kappa_2 \mathbf{E} + (\kappa_1 - \kappa_2) \mathbf{d}_1 \mathbf{d}_1^T = \mathbf{S}$.

3.2.2. A New Method for Umbilical Points Lemma 3.4 and Proposition 3.5 imply not only that Eq.(3) can be simplified by $\mathbf{t}$, but also that it can be applied to any parameterization to free $\mathbf{S}$ from symmetry. Namely, there exists $(x; y) \in [0; 1] \times [0; 1]$ and $x + y \leq 1$ for a triangle $\triangle abc$ such that:

$$x\mathbf{t}_a + y\mathbf{t}_b + (1 - x - y)\mathbf{t}_c = \mathbf{0}$$

We call $\triangle abc$ satisfying Eq.(5) a zero-crossing triangle. A triangle containing an umbilical point is called an umbilical triangle. A zero-crossing triangle is not necessarily an umbilical triangle in practical computation.

We shall continue the numerical analysis of Eq.(5) and illustrate more of its behaviors through experiments later—in Sections 5 and 7, respectively.

4. Finding a Zero-Crossing Triangle

Given a mesh model, numerous methods have been developed to derive the principal curvatures and directions at each vertex [**?**, **?**]. In this section, we develop a local parameterization of high efficiency to synthesize the estimated principal curvatures and principal directions into shape operators consistently in a coordinate system.

4.1. Overview of the Method

Suppose principal curvatures and principal directions—that is, shape operators at their respective frames—at vertices have been acquired by some means, either analytically or numerically. To depict our method, it suffices to carry out Eq.(5) on one triangle $\triangle abc$ of the mesh model. Now imagine a right-handed uvw coordinate system at a point p_0 inside $\triangle abc$ with the w-axis parallel to the normal of $\triangle abc$. We must project shape operators of a, b, and c onto the uv plane, respectively. Once they are determined, interpolation is performed via Eq.(5). The resultant interpolating umbilical point is calculated as $x\mathbf{a} + y\mathbf{b} + (1 - x - y)\mathbf{c}$.

The remaining issues involve projecting relevant shape operators to the parametric domain on $\triangle abc$. We shall portray our method to parameterize a shape operator locally for this purpose in the next subsection.

4.2. Local Parameterization of S

We here illustrate concrete implementations of our method with two local parameterizations.

4.2.1. Local Orthogonal Projection We choose uvw-coordinates so that a point on the surface is at the origin and the three axes are the principal directions and normal, respectively. The surface is then expressible in Monge form as:

$$w = \frac{1}{2}(\kappa_{1u}^2 + \kappa_{2v}^2) + \frac{1}{6}(b_{0u}^3 + 3b_{1u}^{2v} + 3b_{2uv}^2 + b_{3v}^3) + O(\mathbf{u})^4$$

where $O(\mathbf{u})^4$ is a Peano remainder.

For each vertex of the mesh, the Monge form in its neighborhood can be established numerically and uniquely to degree 2 by estimating its normal, principal directions, and principal curvatures from a set of near-neighbor vertices. However, the shape operators of two distinct vertices cannot be put into arithmetical operation directly because they are obtained separately and measured in their respective frames. To reconcile them to a consistent frame, the Monge forms need to be reshaped in a common parametric domain. Given a local xyz coordinate system at p_0 and a Monge form in the local uvw coordinate system at p_k as shown in Fig. 1 [Figure 1: see original paper], we intend to re-parameterize $\mathbf{S}|_{p_k}$ from the frame p_k-uvw to the frame p_0-xyz. As described in Section 3.1, $\mathbf{S}$ is computable by setting $\mathbf{r}(\mathbf{x}) = [\mathbf{x}; z(\mathbf{x})]^T$ if $\mathbf{J}_\mathbf{x}(z)$ and $\mathbf{H}_\mathbf{x}(z)$ are computable, where $\mathbf{J}_\mathbf{x}(z)$ and $\mathbf{H}_\mathbf{x}(z)$ are the Jacobian and Hessian matrices of z in terms of $\mathbf{x}$, respectively.

Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the unit vectors in the frame p_0-xyz along the positive directions of the x, y, and z axes, respectively; $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$, and $\bar{\mathbf{e}}_3$ are those of the u, v, and w axes, respectively, as displayed in Fig. 1. Let $a_{ij} = \mathbf{e}_i \cdot \bar{\mathbf{e}}_j$, $b_i = (p_k - p_0) \cdot \mathbf{e}_i$, $i, j = 1, 2, 3$, and $\mathbf{U} = [u; v; w]^T$, $\mathbf{X} = [x; y; z]^T$. We have $\mathbf{X} = [a_{ij}]_{3\times3}\mathbf{U} + [b_i]_{3\times1}$.

Let $\mathbf{a} = [a_{12}; a_{11}]^T$, $\mathbf{b} = [a_{22}; a_{21}]^T$, $\mathbf{c} = [a_{23}; a_{13}]^T$, $\mathbf{A} = [\mathbf{b}; -\mathbf{a}]$, $\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $\mathbf{K} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$. By elementary computation, we have:

$$\mathbf{J}_\mathbf{x}(\mathbf{u}) \triangleq \mathbf{B}, \quad \mathbf{H}_\mathbf{x}(\mathbf{u}) \triangleq \frac{\det([\mathbf{c}; \mathbf{b}])}{\det(\mathbf{B})^3}\mathbf{A}^T\mathbf{K}\mathbf{A}$$

$$\mathbf{J}_\mathbf{x}(\mathbf{v}) \triangleq \mathbf{B}, \quad \mathbf{H}_\mathbf{x}(\mathbf{v}) \triangleq \frac{\det([\mathbf{a}; \mathbf{c}])}{\det(\mathbf{B})^3}\mathbf{A}^T\mathbf{K}\mathbf{A}$$

$$\mathbf{J}_\mathbf{x}(\mathbf{w}) \triangleq \mathbf{0}, \quad \mathbf{H}_\mathbf{x}(\mathbf{w}) \triangleq \mathbf{B}^T\mathbf{K}\mathbf{B}$$

where $\triangleq$ means the evaluation is taken at p_k (or $\bar{p}_k$), e.g., $\mathbf{J}_\mathbf{x}(\mathbf{w})|_{p_k}$.

According to $z = a_{31}u + a_{32}v + a_{33}w$ in Eq.(7) and the resultant Eqs.(8)-(12), we eventually achieve $\mathbf{J}_\mathbf{x}(z)$ and $\mathbf{H}_\mathbf{x}(z)$ at $\bar{p}_k$, and thus $\mathbf{S}|_{p_k}$.

4.2.2. Local Conformal Transformation Instead of orthogonally projecting them onto a parametric plane, we can flatten the matrices of shape operators onto the triangle plane by transforming the vertex frames to the face frame rotating around the cross product of their respective normals with the plane normal, e.g., $\mathbf{e}_3$ and $\bar{\mathbf{e}}_3$ in Fig.1, while keeping the angles of the principal directions [?]. Only three triangle vertices account for the interpolation, and this local conformal transformation can be regarded as a local structure of a certain global conformal mapping. Our experiments show that it works almost as well as the local orthogonal projection.

5. Error Analysis

We now address the fundamental question of its convergence rate as the mesh resolution increases.

5.1. Invariant in Linear Interpolation

We examine the quantitative behavior of our linear interpolation method on a surface locally represented in the Monge form. Before proceeding, we provide a lemma to this end:

Lemma 5.1. The interpolating point of a zero-crossing triangle is invariant in any curvilinear coordinate arising from a rigid transformation.

Proof. $\mathbf{S}$ is expressed in terms of $\mathbf{u}$. Given a rigid transformation between $\mathbf{u}$ and $\hat{\mathbf{u}}$, i.e., $\mathbf{u} = \mathbf{R}\hat{\mathbf{u}} + \mathbf{u}_0$, where $\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\mathbf{u}_0$ is a constant translation, then $\mathbf{S}$ in terms of $\hat{\mathbf{u}}$ can be represented by $\mathbf{R}^T\mathbf{S}\mathbf{R}$. Elementary computation shows that $\mathbf{t}(\hat{\mathbf{u}}) = \mathbf{M}\mathbf{t}(\mathbf{u})$ where $\mathbf{M} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$. Suppose the three vertices of a zero-crossing triangle are represented as $(\mathbf{a};\mathbf{b};\mathbf{c})$ in terms of $\mathbf{u}$ and $(\hat{\mathbf{a}};\hat{\mathbf{b}};\hat{\mathbf{c}})$ in terms of $\hat{\mathbf{u}}$, respectively. We have $x\mathbf{t}(\hat{\mathbf{a}}) + y\mathbf{t}(\hat{\mathbf{b}}) + (1-x-y)\mathbf{t}(\hat{\mathbf{c}}) = \mathbf{M}(x\mathbf{t}(\mathbf{a}) + y\mathbf{t}(\mathbf{b}) + (1-x-y)\mathbf{t}(\mathbf{c}))$. Because $\det(\mathbf{M}) = 1$, it holds that $x\mathbf{t}(\hat{\mathbf{a}}) + y\mathbf{t}(\hat{\mathbf{b}}) + (1-x-y)\mathbf{t}(\hat{\mathbf{c}}) = \mathbf{0} \Leftrightarrow x\mathbf{t}(\mathbf{a}) + y\mathbf{t}(\mathbf{b}) + (1-x-y)\mathbf{t}(\mathbf{c}) = \mathbf{0}$. We verify this lemma immediately.

5.2. Error Analysis

By substituting the Monge form of Eq.(6), elementary computation yields:

$$\mathbf{S} = \begin{bmatrix} \kappa_1 + b_{0u} + b_{1v} & b_{1u} + b_{2v} \\ b_{1u} + b_{2v} & \kappa_2 + b_{2u} + b_{3v} \end{bmatrix} + O(\mathbf{u})^2$$

Hence:

$$\mathbf{t}(\mathbf{u}) = \begin{bmatrix} \kappa_1 - \kappa_2 + (b_0 - b_2)u + (b_1 - b_3)v \\ 2(b_{1u} + b_{2v}) \end{bmatrix} = \mathbf{t}_0 + u\mathbf{t}_1 + v\mathbf{t}_2 + O(\mathbf{u})^2$$

where $\mathbf{t}_0 = \begin{bmatrix} \kappa_1 - \kappa_2 \\ 0 \end{bmatrix}$, $\mathbf{t}_1 = \begin{bmatrix} b_0 - b_2 \\ 2b_1 \end{bmatrix}$, and $\mathbf{t}_2 = \begin{bmatrix} b_1 - b_3 \\ 2b_2 \end{bmatrix}$. $\mathbf{t}_1$ and $\mathbf{t}_2$ are the principal linear parts.

Given a $\triangle abc$ in the uv domain with origin p, it covers a point $\bar{\mathbf{d}} = [\bar{u}; \bar{v}]^T$ under consideration. Without loss of generality, we can imagine that p is inside $\triangle abc$, for example, the centroid, according to Lemma 5.1. Suppose the barycentric coordinates of $\bar{\mathbf{d}}$ in terms of $\triangle abc$ are $(x; y)$. Denote $h = \max(|\mathbf{a} - p|; |\mathbf{b} - p|; |\mathbf{c} - p|)$. Then:

$$x\mathbf{t}(\mathbf{a}) + y\mathbf{t}(\mathbf{b}) + (1 - x - y)\mathbf{t}(\mathbf{c}) = \mathbf{t}_0 + \bar{u}\mathbf{t}_1 + \bar{v}\mathbf{t}_2 + O(h^2)$$

According to Eqs.(15) and (16), we have:

$$\mathbf{t}(\bar{\mathbf{d}}) = x\mathbf{t}(\mathbf{a}) + y\mathbf{t}(\mathbf{b}) + (1 - x - y)\mathbf{t}(\mathbf{c}) + O(h^2)$$

Eq.(17) reveals the operational principle of our method. It means that $\mathbf{t}$ is approximated by linear interpolation to accuracy of $O(h^2)$. Here h can be relaxed to be the resolution size of the mesh—the maximal length of all edges. It also reflects that the key to linear interpolation is that $\mathbf{t}$ can be linearized locally such that $\mathbf{t}$ is dominated by its linear terms. Either $\mathbf{t}_1 \neq \mathbf{0}$ or $\mathbf{t}_2 \neq \mathbf{0}$ is required to make Eq.(17) operate properly, namely $b_{1b}2 \neq 0 \wedge (b_0 - b_2)(b_1 - b_3) \neq 0$. Eq.(18) is satisfied for a generic umbilical point. If the mesh is fine enough, a zero-crossing triangle of $\triangle abc$ implies that $\triangle abc$ is also umbilical by both Eqs.(17) and (18).

Because an interpolating point can approximate a zero point of $\mathbf{t}$—that is, an umbilical point—we need to address the further issue of what the accuracy is between an umbilical point and an interpolating point to complete our discussion.

Assume $\bar{\mathbf{d}}$ is an interpolating point. By Eq.(16):

$$\mathbf{t}_0 + \bar{u}\mathbf{t}_1 + \bar{v}\mathbf{t}_2 = O(h^2)$$

Suppose $\mathbf{t}(\mathbf{d}) = \mathbf{0}$ inside $\triangle abc$—that is, $\mathbf{d} = [u; v]^T$ is an umbilical point. Then:

$$\mathbf{t}_0 + u\mathbf{t}_1 + v\mathbf{t}_2 = O(h^2)$$

Eq.(20) subtracted from Eq.(19) gives:

$$(\bar{u} - u)\mathbf{t}_1 + (\bar{v} - v)\mathbf{t}_2 = O(h^2)$$

According to the Cauchy-Schwarz inequality, we have:

$$\|\mathbf{d} - \bar{\mathbf{d}}\| = \sqrt{(\bar{u} - u)^2 + (\bar{v} - v)^2} = O(h^2)$$

On the other hand, if the umbilical point is generic and the mesh resolution is high enough, $\mathbf{t}$ is approximated to sufficient accuracy by its principal linear components and thus a zero-crossing triangle coincides with an umbilical triangle. In this case we obviously have $\|\mathbf{d} - \bar{\mathbf{d}}\| \leq O(h)$. Therefore, we have the following proposition:

Proposition 5.2. If the mesh resolution is sufficiently high, the interpolating point of a zero-crossing triangle approximates an umbilical point to accuracy between $O(h)$ and $O(h^2)$.

5.3. Error Propagation

The above analysis is based on exact $\mathbf{S}$. However, principal curvatures and principal directions are estimated in practice only up to some accuracy. Thus, their errors must be considered in the estimation of umbilical points by linearly interpolating $\mathbf{S}$, i.e., error propagation from shape operators to an interpolating point. According to the discussion in Section 5.2, it entirely relies on the accuracy of the estimated $\mathbf{S}$.

We consider it under the conditions presented by [**?**]:

Proposition 5.3. Given the position, gradient, and Hessian of coordinate functions approximated to $O(h^{d+1})$, $O(h^d)$, and $O(h^{d-1})$, respectively, and assuming the condition number of the Jacobian matrix is bounded, we have: (a) The angle between the computed and exact normals is $O(h^d)$; (b) the components of the shape operator and curvature tensor are approximated to $O(h^{d-1})$.

For a mesh model, d is typically 2 such that $\mathbf{S}$ is approximated to accuracy $O(h)$, which gives us the following corollary:

Corollary 5.4. A resultant umbilical point by linear interpolation is typically approximated to accuracy $O(h)$.

6. Reconditioning Umbilical Points

Umbilical points are difficult to recondition with complete fidelity although they are inherent in a surface. Two factors strongly affect their computational quality: the distribution of the $\mathbf{S}$ tensor and geometric detail across the surface. In practical usage, two crucial issues must be addressed: how to judge an umbilical point to be spurious, and how to compute an operable umbilical point set.

Judging spurious umbilical points. To the best of our knowledge, no work has been done for this purpose in the literature yet. We propose a method to determine if an umbilical point is spurious by a necessary condition: An umbilical point is either spherical or planar and thus distributes over an elliptic ($\kappa_G > 0$) or a planar ($\kappa_G = 0$ and $\kappa_H = 0$) region of the surface [**?**]. It generally occurs as an isolated point in the elliptical region; furthermore, a flat region is left out of account since it is unusually of interest in practical applications. Another obvious observation indicates that a significant umbilical point should be supported over a sufficiently large elliptic region; in contrast, local shape variation only yields a small region of the same type. Based on these considerations, our first strategy is devised as follows:

1. First, the surface is partitioned into parts with surface type labels based on κ_G and κ_H. Only elliptic surface types remain, i.e., Peak and Pit types [?], for further treatment.
2. Then, two fundamental operations in mathematical morphology—dilation operator $\mathcal{D}$ and erosion operator $\mathcal{E}$—are used in turn. They compose an opening operator $\mathcal{D}_n \circ \mathcal{E}_n$ with n erosions followed by n dilations, applied to the elliptic parts to erase narrow connectors and open up large holes or caves. n is determined by the sizes of narrow connectors to be erased. Umbilical points are eroded away near the boundaries of elliptic parts.
3. Finally, the remaining umbilical points are kept over the elliptic parts. Inconsequential umbilical points can be cleaned out significantly in this way, as we will demonstrate in our experiments.

Computing operable umbilical points. A real-world mesh model usually contains abundant curvature variation across its surface arising from various desired or undesired factors, such as detail enhancement and sampling noise. The sensitivity of umbilical points to these variations reflects their characteristic response to shape undulation, leading to a massive number of umbilical points. However, too many umbilical points might affect their practical application. To make them operable, their number should be drastically reduced. For example, they are reduced to simplify the topology of the extracted field to cut a surface for quadrangulation [?]. In this scenario, an approach to smoothing or denoising shape operators is beneficial for minimizing the number to meet different requirements—such as removing superfluous points by C^∞ smoothing [?] or by refinement of principal frame fields [?]. In contrast to those methods that denoise second-order differential quantities, we smooth normals by a linear filter and then estimate shape operators based on the resultant normals. Our experimental results will show that our method can decimate undesired umbilical points effectively.

7. Experimental Results and Discussion

In this section, we demonstrate our method through experiments and comparisons with other methods based on a surface of analytical representation and some real-world mesh models.

7.1. Method Validation

We confirm our proposed techniques using a practical example with known umbilical points on a given Monge patch:

$$z = 0.5(x^2 + y^2) + 1.2x^3 - 0.5x^{2y} + 0.2xy^2 + 1.6y^3$$

Since Eq.(21) is a polynomial expression, all its umbilical points can be found by solving a polynomial system, as done in [?]. In fact, there are exactly three umbilical points on the patch:

$$\mathbf{p}_1 = [0; 0; 0]^T$$
$$\mathbf{p}_2 = [-0.95695088; -0.18571175; -0.50829269]^T$$
$$\mathbf{p}_3 = [-0.39165274; -0.81922023; -0.52924773]^T$$

Given a set of 5.4K points by random sampling inside the region $[-1.2; 1.2] \times [-1.2; 1.2]$ on the parametric plane, a Delaunay triangulation is constructed. We polygonize this patch into a triangular mesh using the Delaunay triangulation. For clarity, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are marked noticeably by their respective bounding boxes as in Fig. 2 [Figure 2: see original paper].

We devise the experiments following the progression from fine to coarse shape operators.

7.1.1. Numerical Behavior We first consider an instance excluding the influence of noise. The exact principal curvatures and principal directions are both computed analytically at each vertex according to Eq.(21), so shape operators are ready to be applied to our approach regardless of numerical errors in discrete estimation. The linearly interpolating points are displayed in Fig. 4 [Figure 4: see original paper]. As expected, it exhibits favorable interpolation despite the irregular mesh: three umbilical triangles are identified correctly and the interpolating points are close to the ground-truth umbilical points with sufficient accuracy.

We have shown that any valid parameterization can be applied to the shape-operator-based interpolation. Besides local orthogonal projection, we also provide a local conformal transformation as a means of implementing the method in Section 4.2.2. Tables 1 and 2 list the experimental data for the two implementations. There is no appreciable difference between them, as also expected in our additional experiments. Therefore, shape-operator-based interpolation via orthogonal projection is used exclusively in the following comparison experiments.

7.1.2. Comparisons Can other comparable methods perform better than our proposed method, since ours might fail to respect the true umbilical points as shown in Fig. 5(d) [Figure 5: see original paper]? In this subsection, we compare our shape-operator-based interpolation method to the index-based method grounded on Eq.(2) and the curvature-tensor-based interpolation method via global conformal parameterization, denoted as S, I, and C methods for short, respectively. For curvature-tensor interpolation, we apply LSCM parameterization from [?] to perform the global parameterization. Because it requires the mesh to be homeomorphic to a planar disk or a planar square in the parametric domain, the Monge patch of Eq.(21) happens to satisfy this requirement. In this scenario, we can re-flatten it to the planar domain of conformal parameterization.

Reusing the shape operators from Section 7.1.1, both analytical and discrete, we compute the umbilical points with the three methods, respectively. Figs. 2 and 4 illustrate the resultant interpolating points in the analytical case, as do Figs. 2 and 5 in the discrete case. It is obvious that the interpolation behavior of C and S methods are quite similar—something that is expected. Without regard to the error effects of global discrete parameterization, their resultant umbilical points can be considered comparable; unfortunately, the error of approximate global parameterization can make the interpolation less reliable, which will be shown by experiments in the following text. In vivid contrast to both C and S, the I method yields a cluster of points near an umbilical point rather than a single point.

Fig. 5 makes clear that both I and C methods cannot outperform S. These facts become more evident in the following experiments.

7.2. Linear Interpolation over Mesh Models

We demonstrate our method with real-world mesh models. Fig. 6 [Figure 6: see original paper] shows the resultant umbilical points on a hand model by the three methods, respectively. To help understand their distribution over the mesh, one branch of estimated principal directions is also displayed. In the I method, a cluster of vertices can be detected arising from an umbilical point, typically either 1, 2, or 3 vertices (see Fig. 6(a)). The C method fails to work near the tips of the fingers except for the thumb because the parameter gradients of the underlying global conformal parameterization are too large (see Fig. 6(b)). The S method behaves the best (see Fig. 6(c)). Of course, the S method works even better if the model is decomposed into smaller charts and then flattened to reduce distortion.

The second example is a rocker-arm model. It is closed and thus global parameterization cannot be performed. Fig. 7 [Figure 7: see original paper] illustrates the difference between the S method and the I method. The former usually determines only one umbilical point near a singular point in the principal direction field, but the latter does not. Because sufficient points per boundary curve are required to estimate the index, the index-based method is more sensitive to the irregularity of the mesh.

7.3. Filtering Umbilical Points

Given trivially-estimated shape operators, the previous example with ground-truth umbilical points exhibits some spurious ones even if the mesh is noise-free (see Figs. 2 and 5(d)). As pointed out in Section 6, two strategies can be exploited to recondition the resultant umbilical points and thus increase their practical utility.

The first strategy applies the opening operator to an elliptical region over the surface to erase narrow connectors usually related to geometry details or transition regions between different surface types. The larger n is, the more umbilical points are removed. To balance the fidelity and practicability of resultant umbilical points, $n = 2$ is chosen in our experiments. Furthermore, flat regions are left out of account since they are unusually of interest in practical applications. We apply this strategy to Fig. 5(d), and the result is shown in Fig. 8 [Figure 8: see original paper]. We can see that the spurious points outside elliptical regions are eliminated. Fig. 6(d) shows the reconditioned umbilical points for the hand model.

However, the first strategy is not capable of handling complex geometry, for instance, the Stanford Bunny with abundant curvature variation. Fig. 9 [Figure 9: see original paper] shows the interpolating umbilical points handled without and with the opening operator, respectively. Because the "low" mesh resolution of the Bunny cannot accommodate its surface roughness, too many umbilical points still remain even after opening operation, which affects their practical utility. In this setting, an approach to smoothing or denoising shape operators is beneficial for further improvement of filtering quality. As an alternative, we apply linear fitting to the normals before shape operators are estimated. The experimental results are shown in Fig. 10 [Figure 10: see original paper]. This strategy works very well for the rocker-arm model; as illustrated in Fig. 11 [Figure 11: see original paper], it makes a significant improvement in contrast to Fig. 7.

Fig. 12 [Figure 12: see original paper] results from the combination of the two strategies. It shows that the resultant points reflect symmetry for a symmetrical shape. They can be used as anchor points to extract the symmetrical plane of the Torso.

7.4. Computational Cost

Given principal curvatures and principal directions of vertices over a mesh, linear interpolation of shape operators for umbilical points is performed by iterating over all triangles. For each triangle interpolation, no optimization computation is required. Instead, only some necessary floating-point arithmetic for local reparameterization of shape operators onto each triangle plane (as described in Section 4.2) is needed. Therefore, the computational cost of our algorithm increases linearly with the number of triangles and runs on-the-fly, as opposed to the cost for estimating shape operators.

8. Conclusion

In this paper, we described a novel approach for umbilical-point detection over a mesh by interpolating shape operators via local parameterization. The shape operators do not need to be symmetrical anymore, and we propose using a local parameterization by orthogonal projection or local conformal transformation to cooperate with the interpolation of shape operators. Furthermore, we conducted a systematic error analysis of the interpolation to demonstrate its efficiency.

Linear interpolation is superior to the index-based method because the latter must collect sufficient neighboring points for index computation and, accordingly, the former is less sensitive to the regularity of the underlying mesh. In contrast to curvature-tensor-based methods relying on global conformal parametrization, the proposed method is significantly advanced because its local parametrization is more robust and introduces no additional errors in parameterization. Furthermore, it does not suffer from the topological requirements of the mesh surface for global parametrization, thus removing significant computational burden and implementation complexity without constructing a global parametrization or segmenting a complex model. As a result, our method saves considerable time and is more flexible and easier to implement.

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Note: Figure translations are in progress. See original paper for figures.