Microwave Interferometer Measurement and Inversion Methods for Plume Electron Density (Postprint)

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Abstract

Research on inversion methods for thruster plume density measurement using microwave interferometers. Focusing on numerical solution of the Abel inverse transform for plume plasma density inversion, four methods including the discretization algorithm, cubic spline interpolation algorithm, Hankel-Fourier algorithm, and Nestor-Olsen algorithm are employed for simulation analysis and comparison. Through comparison of inversion errors of these four methods under both noise-free and noisy conditions, methods suitable for actual interferometer data processing are identified; through analysis of the relationship between sampling points and inversion, methods and numerical references for solving undersampling problems are provided; based on the analysis and comparison, the necessary steps for numerical processing are summarized. Validation is performed using simulated experimental test data and the inversion results are analyzed.

Full Text

Study on Inversion Methods for Plume Electron Density Measurement by Microwave Interferometry

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Abstract: Four different numerical Abel inversion algorithms for the retrieval of plume density profile by microwave interferometry, including the discretization method, the cubic spline interpolation method, the Hankel-Fourier transform method and the Nestor-Olsen method, are compared for the cases with and without noise. After analysis of the effect of sampling density on the retrieval error, the smoothing and interpolating solution is proposed to mitigate the under-sampling related problem. As a summary, the procedures of the retrieval and processing are proposed. Experimental data are used to verify the proposed method and the analysis of the results are provided.

Keywords: Abel inversion; interferometry; plume diagnostic; Hankel-Fourier method

Microwave interferometry serves as an important diagnostic tool for measuring electron density in thruster plumes. The technique works by measuring the phase change of microwave signals as they propagate through the plasma plume, yielding path-integrated values of plasma electron density that can then be inverted to obtain the spatial distribution. When the electron density distribution in the plume can be approximated as axisymmetric, Abel inversion enables recovery of the radial density profile.

In experimental practice, data processing for microwave interferometer measurements is often simplified. Cappelli et al. directly calculated the line-integrated electron density from measured values [**?**], while James employed an analytical approach using Gaussian function fitting prior to Abel inversion [**?**]. However, with advances in thruster technology, modern plumes have diameters as small as the centimeter scale, and undersampling can occur during measurement, complicating accurate inversion of plasma electron density. To address these challenges, this paper examines four numerical Abel inversion methods—discretization, cubic spline interpolation, Hankel-Fourier transform, and Nestor-Olsen—analyzing their performance under various noise conditions and sampling densities, and validates the approaches using simulated experimental data.

1.1 Theory of Plume Density Inversion

Based on electromagnetic wave propagation theory in plasmas [**?**, **?**], the plasma refractive index is a function of electron density and magnetic field when particle thermal motion is neglected. The electron density determines the plasma frequency $\omega_{pe} = \sqrt{n_e e^2/(m_e \epsilon_0)}$, while the magnetic field determines the electron cyclotron frequency $\omega_{ce} = eB/(m_e c)$, where n_e is plasma electron density, m_e is electron mass, ϵ_0 is vacuum permittivity, e is electron charge, B is magnetic field strength, and c is the speed of light.

When the electromagnetic wave frequency ω satisfies $\omega_{pe}, \omega_{ce} \ll \omega \ll \omega_\mu$, the plasma can be treated as a non-magnetized cold plasma, allowing neglect of thermal particle motion and magnetic field effects on wave propagation. The plasma refractive index n can then be expressed as:

$$n = \sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}} \approx 1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2} = 1 - \frac{n_e}{2n_c}$$

where n_c is the critical electron density:

$$n_c = \frac{m_e \epsilon_0 \omega^2}{e^2} = 1.24 \times 10^{-2} f^2$$

When an electromagnetic wave traverses a plasma along path L, the phase shift measured by microwave interferometry represents the integral of phase delay along the propagation path:

$$\Delta\phi = -\frac{2\pi}{\lambda_0} \int_L (n-1) dl = \frac{\pi}{\lambda_0 n_c} \int_L n_e dl$$

Most plasma plumes exhibit cylindrical symmetry, meaning the electron density distribution can be expressed as a function of radial coordinate r alone, $f(r)$. By scanning along different paths, the total phase delay for various line-of-sight positions can be obtained, enabling reconstruction of the radial electron density distribution through Abel inversion.

1.2 Abel Inverse Transform

For a radially symmetric physical quantity $f(r)$ with radius R, integrating $f(r)$ along a chord parallel to the y-axis at coordinate x yields the chord-integrated quantity:

$$F(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f(r) dy$$

This $F(x)$ represents the interferometric phase delay at position x. Transforming the integration variable to r gives the Abel integral:

$$F(x) = 2 \int_x^R \frac{f(r) r dr}{\sqrt{r^2 - x^2}}$$

The corresponding Abel inverse transform is:

$$f(r) = -\frac{1}{\pi} \int_r^R \frac{dF/dx}{\sqrt{x^2 - r^2}} dx$$

[Figure 1: see original paper]

2 Numerical Solution Methods for Abel Inverse Transform

Abel inversion solutions can be categorized as analytical or numerical. Analytical methods fit measured data with a priori distribution functions before solving, while numerical methods convert the integral operation into discrete summation, offering faster computation suitable for practical microwave interferometer data processing.

Equation (6) presents two main challenges: a singularity at $x = r$ and the presence of a derivative term in the integrand. This paper focuses on four numerical approaches: discretization, cubic spline interpolation, Hankel-Fourier transform, and Nestor-Olsen methods.

For consistent comparison, we assume the microwave interferometer provides $2N + 1$ data points over $[-R, R]$, denoted as $F_i = F(x_i)$ with $x_i = i\Delta x$ for $i = 0, \pm 1, \dots, \pm N$, where Δx is the measurement step. The radial grid for $f(r)$ matches the x-grid: $r_i = i\Delta r$ with $\Delta r = \Delta x$ and $f_i = f(r_i)$ for $i = 0, \dots, N$.

2.1 Discretization Method

The discretization method is the simplest Abel inversion algorithm. It divides the integration domain into intervals between adjacent data points and shifts the x-axis grid by $\Delta x/4$ to avoid the singularity. The derivative term in equation (6) is approximated from neighboring data points [?]. The discretization formula is:

$$f_i = \sum_{j=i}^{N-1} A_{ij} \frac{F_{j+1} - F_j}{\Delta x}$$

where the coefficients A_{ij} are given by:

$$A_{ij} = \begin{cases} \frac{1}{\pi}\left[\sqrt{(j+1)^2 - i^2} - \sqrt{j^2 - i^2}\right] & \text{for } j \geq i+1 \\ \frac{1}{\pi}\sqrt{(i+1)^2 - i^2} & \text{for } j = i \end{cases}$$

2.2 Cubic Spline Interpolation Method

In the discretization method, derivatives are calculated from adjacent measurement values, which introduces significant error when measurement steps are large or data points are sparse, even in noise-free conditions. To obtain more accurate derivative estimates, the $2N + 1$ measurement points can be divided into $2N$ intervals, with a cubic spline function constructed from $2N$ cubic polynomial segments to approximate the data before differentiation [?].

2.3 Hankel-Fourier Algorithm

Abel, Fourier, and Hankel transforms form a cyclic relationship known as the FHA ring theory [?]: applying an Abel transform followed by a Fourier transform yields the same result as applying a Hankel transform. Denoting the transform operators as A, F, and H respectively:

$$FA = H$$

The Fourier and Hankel transform pairs are:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx$$
$$F^{-1}[f(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{j\omega x}d\omega$$

$$H_n[f(x)] = \int_0^{\infty} f(x)J_n(kx)xdx$$
$$H_n^{-1}[f(k)] = \int_0^{\infty} f(k)J_n(kx)kdk$$

where J_n denotes the n-th order Bessel function.

From equation (8), the radial distribution function $f(r)$ can be obtained by:

$$f(r) = H_0^{-1}F[A[f(r)]]$$

Since $A[f(r)]$ corresponds to the interferometer measurement $F(x)$, we first apply a Fourier transform to $F(x)$, then perform a zeroth-order Hankel transform:

$$f(r) = \int_0^{\infty} \left[\int_{-\infty}^{\infty} F(x)e^{-j\omega x}dx\right] J_0(\omega r)\omega d\omega$$

where J_0 is the zeroth-order Bessel function. For discrete measurements [?]:

$$f(r_i) = \frac{1}{2N+1}\sum_{k=0}^{N-1}\left[\sum_{j=-N}^{N} F(x_j)\cos\left(\frac{2\pi kj}{2N+1}\right)\right] J_0\left(\frac{2\pi ki}{2N+1}\right)$$

2.4 Nestor-Olsen Algorithm

Nestor and Olsen (1960) proposed a numerical solution for $f(r)$ [?]:

$$f(r_i) = \frac{1}{\pi}\sum_{j=i}^{N-1} B_{ij}F(x_j)$$

where the coefficients B_{ij} are:

$$B_{ij} = \begin{cases} A_{ij} - A_{i,j-1} & \text{for } j \geq i+1 \\ A_{ii} & \text{for } j = i \end{cases}$$

3 Simulation Analysis

To analyze and compare these methods, we first select the test function:

$$f(r) = \frac{1}{\pi(1+r^2)}, \quad -1 \leq r \leq 1$$

The corresponding Abel transform for this function is:

$$F(x) = 1 - \frac{1}{\pi} \arctan\left(\frac{x}{\sqrt{1-x^2}}\right), \quad -1 \leq x \leq 1$$

3.1 Determination of Grid Offset

Both the discretization and cubic spline algorithms described above avoid the singularity at $x = r$ by shifting the x-axis grid. To determine the optimal offset, various Δx values were tested. The results are shown in Figure 2 [Figure 2: see original paper].

The calculations reveal that results at $r = 0$ are lower than the true values but converge toward the theoretical solution as r increases. The offset $\Delta x = 4dx$ yields the best performance. All subsequent results for the discretization and cubic spline algorithms use $\Delta x = 4dx$.

3.2 Comparison of Four Algorithms

Using $N = 30, 50, 100, 150$, the four algorithms produce the results shown in Figure 3 [Figure 3: see original paper]. Overall, all methods improve as N increases. For these calculations, the cubic spline algorithm matches the theoretical values most closely, showing noticeable error only near $r = 0$ due to grid shifting. This accuracy stems from using noise-free samples where cubic spline interpolation provides high-precision derivative estimates.

The Hankel-Fourier algorithm also performs well. While error is significant at $N = 30$, results converge toward theoretical values as N increases. Notably, in actual plume diagnostics, the central density peak represents a critical thruster parameter. For $N = 50, 100, 150$, the Hankel-Fourier algorithm provides more accurate central distribution than the other three methods.

The Nestor-Olsen algorithm exhibits large errors near $r = 0$ but approaches theoretical values as r increases. The discretization method is computationally simple but yields larger errors overall.

3.3 Influence of Data Point Count

Figure 3 demonstrates that all four methods improve with increasing N. To quantify this improvement, the root-mean-square error (RMSE) was calculated for the Hankel-Fourier, Nestor-Olsen, and discretization methods at various N values:

$$\sigma = \sqrt{\frac{1}{N+1}\sum_{i=0}^{N}[f_{\text{calc}}(r_i) - f_{\text{theory}}(r_i)]^2}$$

The relationship between RMSE and N is shown in Figure 4 [Figure 4: see original paper]. The Hankel-Fourier algorithm achieves the smallest error, followed by Nestor-Olsen, with the discretization method showing the largest error. For $N < 50$, increasing N significantly improves all three algorithms. However, for $N > 100$, the marginal improvement does not justify the increased computational cost.

When measuring small-scale thruster plumes, limited positioning accuracy of the measurement apparatus may result in sparse sampling, causing large inversion errors. Therefore, before performing Abel inversion, interpolation of sampling points should be considered based on Figure 4 to enhance accuracy.

3.4 Influence of Noise

The preceding analysis assumes ideal noise-free conditions. In practice, noise degrades each method differently. With $N = 80$ and 5% noise added to the true phase shift values, the inversion results appear in Figure 5 [Figure 5: see original paper].

The Hankel-Fourier algorithm demonstrates superior robustness under noisy conditions. In contrast, methods relying on differences between adjacent points or derivatives, such as the discretization and cubic spline methods, exhibit dramatic performance degradation. Consequently, smoothing of measured data is essential. Using the Savitzky-Golay filter (scipy.signal.savgol_filter) on the noisy data before Abel inversion yields improved results, as shown in Figure 6 [Figure 6: see original paper].

3.5 Influence of Radial Distribution Function Discontinuity

For discontinuous radial distributions such as:

$$f(r) = \begin{cases} 1, & r_1 \le r \le r_2 \\ 0, & \text{otherwise} \end{cases}$$

with $r_1 = 25$, $r_2 = 40$, and $\Delta x = 0.1$, the Abel transform is:

$$F(x) = \begin{cases} 2\sqrt{r_2^2 - x^2} - 2\sqrt{r_1^2 - x^2}, & |x| \le r_1 \\ 2\sqrt{r_2^2 - x^2}, & r_1 < |x| \le r_2 \end{cases}$$

Inversion results are shown in Figure 7 [Figure 7: see original paper]. Among the four algorithms, the Hankel-Fourier method produces the largest error for such discontinuous functions, indicating its unsuitability for data with steep variations. Therefore, actual inversion requires careful analysis of plume electron density characteristics before selecting an appropriate method.

Experimental Validation

A simulated microwave interferometer experiment was conducted to validate these methods. PVC-U and rubber cotton pipes were placed on a horizontally movable platform. A pair of conical horn lens antennas served as transmitter and receiver, with a vector network analyzer measuring phase changes through the test objects, simulating the interferometer operation. The experimental setup is shown in Figure 8 [Figure 8: see original paper], with parameters listed in Table 1 .

A 75mm-diameter PVC-U pipe wrapped with 20mm-thick rubber cotton was measured. Assuming uniform, cylindrically symmetric material properties and unit phase change per unit length along the Abel integration path, the corresponding Abel transform is $F(x)$. The normalization coefficient is $k = \max[F(x)]$.

Phase data at 27.45 GHz were selected. Using the phase measured without the pipe as reference, the phase shift at each position is shown in Figure 9 [Figure 9: see original paper]. Due to experimental limitations, the data exhibit asymmetry and center offset. Before inversion, center correction was applied, followed by averaging symmetric points about the center to obtain symmetric data. Accounting for antenna spot size, deconvolution was performed based on the known beam energy distribution before Abel inversion. Results were multiplied by $1/k$ to obtain Figure 10 [Figure 10: see original paper].

All four methods approximately recover the test object' s geometry, but significant errors occur at the center. Error sources include: (1) electromagnetic properties of the pipe material—Figure 11 [Figure 11: see original paper] compares corrected measurements with theoretical data, showing measured phase changes at the center are smaller than theoretical values, a phenomenon observed across frequencies and more pronounced at some frequencies, related to wave propagation within the pipe; (2) measurement errors; (3) deconvolution errors; and (4) inversion algorithm errors.

5 Analysis and Conclusions

To address the needs of microwave interferometer data processing and inversion for thruster plume plasma electron density distribution, this paper performed simulation analysis and comparison of four numerical methods: discretization, cubic spline interpolation, Hankel-Fourier transform, and Nestor-Olsen. The study examined the effects of sampling density and measurement noise on inversion accuracy and applied the methods to simulated experimental data.

Numerical and experimental results demonstrate that for smoothly varying data, the Hankel-Fourier algorithm achieves high accuracy, particularly in determining the density peak with minimal error. Under noisy conditions, it exhibits superior robustness compared to other methods. However, it performs poorly for data with steep variations. Since actual plume density distributions resemble Gaussian profiles without the steep edges present in our test case, the Hankel-Fourier method is well-suited for microwave interferometer data inversion.

Practical microwave interferometer data processing should include: (1) data symmetrization, (2) smoothing, and (3) interpolation if undersampling occurs to reduce inversion errors. When selecting sampling or interpolation points, the results presented here provide a reference for balancing computational speed and accuracy.

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Note: Figure translations are in progress. See original paper for figures.