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Authors: Zeng, Xiao-Xiong, Liu, Xian-Ming, Li, Li-Fang

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Abstract

With the non-local observables such as two point correlation function and holographic entanglement entropy, we probe the phase structure of the Born-Infeld-anti-de Sitter black holes. For the case bQ > 0.5, where b is the Born-Infeld parameter and Q is the charge of the black hole, the phase structure is found to be similar to that of the Van der Waals phase transition, namely the black hole undergoes a first order phase transition and a second order phase transition before it reaches a stable phase. While for the case bQ < 0.5, a new phase branch emerges besides the Van der Waals phase transition. For the first order phase transition, the equal area law is checked, and for the second order phase transition, the critical exponent of the heat capacity is obtained. All these results are found to be the same as that observed in the entropy-temperature plane.

Full Text

Phase Structure of Born-Infeld-anti-de Sitter Black Holes Probed by Non-local Observables

Xiao-Xiong Zeng[1],[2], Xian-Ming Liu[3], , Li-Fang Li

[1]School of Material Science and Engineering, Chongqing Jiaotong University, Chongqing 400074, China
[2]Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[3]Center for Theoretical Physics, School of Sciences, Hubei University for Nationalities, Enshi, Hubei 445000, China
State Key Laboratory of Space Weather, Center for Space Science and Applied Research, Chinese Academy of Sciences, Beijing 100190, China

Abstract: Using non-local observables such as the two-point correlation function and holographic entanglement entropy, we probe the phase structure of Born-Infeld-anti-de Sitter black holes. For the case $bQ > 0.5$, we find the phase structure resembles that of the Reissner-Nordström-AdS black hole, where the black hole undergoes a Hawking-Page phase transition, a first-order phase transition, and a second-order phase transition. For $bQ < 0.5$, we discover a new branch for infinitesimally small black holes, leading to a pseudo phase transition in addition to the original first-order phase transition. For both the first-order phase transition and the pseudo phase transition, we verify the equal area law, and for the second-order phase transition, we obtain the critical exponent of the analogous heat capacity near the critical points. All results demonstrate that the phase structure of non-local observables matches that of thermal entropy regardless of the boundary region size in the field theory.

1. Introduction

Investigations into phase transitions in AdS spacetime have attracted considerable attention from theoretical physicists recently. The primary motivation stems from the AdS/CFT correspondence [?, ?, ?], which relates a black hole in AdS spacetime to a thermal system without gravity. In this framework, interesting but intractable phenomena in strongly coupled systems become tractable in the bulk. To probe these fascinating phenomena in field theory, one should employ non-local observables such as two-point correlation functions, Wilson loops, and holographic entanglement entropy, which are dual to geodesic length, minimal area surfaces, and minimal surface area in the bulk, respectively. These observables have been shown to probe non-equilibrium thermalization behavior [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?], superconducting phase transitions [?, ?, ?, ?, ?, ?, ?, ?], and cosmological singularities [?, ?].

In this paper, we employ non-local observables to probe the phase structure of Born-Infeld-anti-de Sitter black holes. Typically, a black hole' s phase structure is understood from a thermodynamic perspective. For an uncharged AdS black hole, a phase transition exists between thermal gas in AdS space and Schwarzschild-AdS black holes [?], later interpreted as confinement/deconfinement phase transition in the dual gauge field theory [?]. When charge is introduced, the AdS black hole undergoes a Van der Waals-like phase transition before reaching a stable state in the entropy-temperature plane [?]. Specifically, there exists a critical charge: for black hole charge smaller than the critical value, the black hole undergoes a first-order phase transition; at the critical charge, the transition becomes second-order; when charge exceeds the critical value, no phase transition occurs and the black hole remains always stable. This Van der Waals-like behavior appears in many contexts: in [?], a 5-dimensional neutral Gauss-Bonnet black hole exhibits Van der Waals-like phase transition in the $T\alpha$ plane (where T is Hawking temperature and α is the Gauss-Bonnet coupling parameter); in [?], similar behavior appears in the $Q\Phi$ plane (where Q is electric charge and Φ is chemical potential). Treating the negative cosmological constant as pressure P and its conjugate as thermodynamic volume V, Van der Waals-like phase transitions have also been observed in the PV plane [?, ?, ?, ?, ?, ?, ?, ?, ?].

Recently, using entanglement entropy as a probe, [?] investigated the phase structure of Reissner-Nordström-AdS black holes and found Van der Waals-like phase transitions in the entanglement entropy-temperature plane. They also obtained the critical exponent of heat capacity near critical points for second-order phase transitions. To further confirm the similarity between thermal entropy and entanglement entropy phase structures, [?] verified the equal area law for first-order phase transitions in the entanglement entropy-temperature plane. Reference [?] has since been generalized to extended spacetime [?], massive gravity [?], and Weyl gravity [?], with all results showing that entanglement entropy exhibits the same phase structure as thermal entropy.

In this paper, besides entanglement entropy, we employ the two-point correlation function to probe black hole phase structure. We choose Born-Infeld-anti-de Sitter black holes as our gravity model, which solves the Einstein-Born-Infeld action. Many works have studied the phase structure of Born-Infeld-anti-de Sitter black holes [?, ?, ?, ?], showing that both the Born-Infeld parameter b and charge Q affect the phase structure, with the condition $bQ > 0.5$ required for non-extremal black hole existence. We find that for $bQ > 0.5$, the phase structure resembles that of Reissner-Nordström-AdS black holes in the entropy-temperature plane. For $bQ < 0.5$, we observe a novel phase structure not previously seen: a new branch emerges compared to Reissner-Nordström-AdS black holes, creating two unstable regions and correspondingly two phase transition temperatures. We probe all these phase structures using non-local observables—two-point correlation function and holographic entanglement entropy—and find they exhibit the same phase structure as thermal entropy.

Our paper is organized as follows. In Section 2, we review the thermodynamic properties of Born-Infeld-anti-de Sitter black holes and study their phase structure in the T-S plane for a fixed charge ensemble. In Section 3, we employ the two-point correlation function and holographic entanglement entropy to probe the phase structure. In each subsection, we check the equal area law and obtain the critical exponent of analogous heat capacity. The final section presents discussions and conclusions.

2.1 Review of the Born-Infeld-anti-de Sitter Black Hole

The 4-dimensional Born-Infeld AdS black hole solves the following action [?]:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - 2\Lambda + \mathcal{L}(F)]$$

where $\mathcal{L}(F) = 4b^2\left(1-\sqrt{1+\frac{F}{2b^2}}\right)$, $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, R is scalar curvature, G is gravitational constant, $\Lambda = -3/l^2$ is the cosmological constant related to AdS radius l, and b is the Born-Infeld parameter related to string tension α' as $b = 1/(2\pi\alpha')$.

Explicitly, the Born-Infeld AdS black hole solution can be written as [?, ?]:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} + \frac{2b^2r^2}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r^4}}\right) + \frac{2Q^2}{3r^2}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{Q^2}{b^2r^4}\right) \tag{2.3}$$

Here ${}_2F_1$ is the hypergeometric function. From Eq. (2.3), we see that in the limit $b \to \infty$, the solution reduces to the Reissner-Nordström-AdS black hole, and in the limit $Q \to 0$, it reduces to the Schwarzschild-AdS black hole.

The ADM mass of the black hole, defined by $f(r_+) = 0$, is given by:

$$M = \frac{r_+}{2} + \frac{r_+^3}{2l^2} + \frac{b^2r_+^3}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r_+^4}}\right) + \frac{Q^2}{3r_+}\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{Q^2}{b^2r_+^4}\right) \tag{2.4}$$

where r_+ is the event horizon radius. The Hawking temperature is:

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left[1 + \frac{3r_+^2}{l^2} + b^2r_+^2\left(1-\sqrt{1+\frac{Q^2}{b^2r_+^4}}\right)\right] \tag{2.5}$$

where we have used Eq. (2.4). Additionally, from the Bekenstein-Hawking entropy-area relation, we obtain the black hole entropy:

$$S = \pi r_+^2 \tag{2.6}$$

Substituting Eq. (2.6) into Eq. (2.5) yields the entropy-temperature relation:

$$T = -\frac{1}{4\pi^{3/2}l^2\sqrt{S}} + \frac{3\sqrt{S}}{4\pi l^2} + \frac{b^2S}{2\pi^{3/2}\sqrt{S}}\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right) \tag{2.7}$$

Clearly, besides charge Q, the Born-Infeld parameter b also affects the phase structure in the T-S plane.

2.2 Phase Transition of Thermal Entropy

Based on Eq. (2.7), we discuss the phase structure of Born-Infeld-anti-de Sitter black holes, focusing on how b or Q influences it. For fixed charge Q, the effect of b on phase structure is shown in Figure 1. For small b, the phase structure resembles the Schwarzschild-AdS black hole, while for large b, it resembles the Reissner-Nordström-AdS black hole as expected. From Figure 1(b), we see that as b increases, the minimum temperature decreases and the entropy (and thus event horizon radius) becomes smaller. That is, the Born-Infeld parameter b promotes AdS black hole formation. More interestingly, for smaller charge, we find a novel phase transition labeled by the red solid line in Figure 1(a). This curve shows two unstable regions, meaning the black hole undergoes the transition sequence: unstable → stable → unstable → stable. Compared to Reissner-Nordström-AdS black holes, a new branch emerges at the onset of phase transition.

The influence of charge on phase structure for fixed Born-Infeld parameter is plotted in Figure 2. The phase structure resembles the Schwarzschild-AdS black hole for small charges and the Reissner-Nordström-AdS black hole for large charges. From Figure 2(a), we observe that larger charge leads to lower minimum temperature. In other words, charge promotes AdS black hole formation, having the same effect as the Born-Infeld parameter. For the case $b = 4$ in Figure 2(b), we see that for some intermediate charge values, a similar phase structure appears as in Figure 1(b), labeled by the red solid line in Figure 2(b). However, for large b, this novel phase structure disappears, as shown in Figure 3. In this case, the Born-Infeld parameter is large enough that the phase structure completely resembles that of the Reissner-Nordström-AdS black hole.

Next, we study in detail the phase structure for cases $b = 4$ and $b = 5$. To do this, we first find the critical charge for a fixed Born-Infeld parameter using the condition:

$$\left(\frac{\partial T}{\partial S}\right)_Q = \left(\frac{\partial^2 T}{\partial S^2}\right)_Q = 0 \tag{2.8}$$

However, from Eq. (2.5), we find it difficult to obtain analytical results directly. Taking $b = 5$ as an example, we show how to obtain it numerically. We first plot a series of curves for different charges in the T-S plane, shown in Figure 3(a), and read off the charge satisfying $(\partial T/\partial S)_Q \approx 0$. With this approximate value, we plot more curves in the T-S plane with smaller steps to refine the charge value. From Figure 3(b), we find the critical charge should be $0.168 < Q_c < 0.17$, labeled by red dashed lines. Finally, we adjust Q_c manually to find the unique solution S_c satisfying Eq. (2.8), yielding $Q_c = 0.168678344129$, $S_c = 0.510691$. Substituting these critical values into Eq. (2.5) gives the critical temperature $T_c = 0.259444$.

Having obtained the critical charge, we plot isocharge curves for $b = 5$ in the

T-S plane, shown in Figure 4. We observe that the black hole undergoes both Hawking-Page and Van der Waals-like phase transitions. Specifically, for $Q = 0$, there is a minimum temperature $T_0 = \sqrt{3}/2\pi$ [?], indicated by the red dashed line in Figure 4(a). When temperature exceeds T_0, two additional black hole branches appear: the small branch is unstable while the large branch is stable. The Hawking-Page phase transition occurs at temperature $T_1 = 1/\pi$ [?], indicated by the red dotted line. This transition also appears in the F-T plane, where F is the Helmholtz free energy defined by:

$$F = M - TS = \frac{r_+}{2} - \frac{r_+^3}{2l^2} + \frac{b^2 r_+^3}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}}\right) + \frac{2Q^2}{3r_+}\,{}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{Q^2}{b^2 r_+^4}\right) \tag{2.9}$$

Note that we implicitly choose pure AdS as the reference spacetime since the free energy vanishes for pure AdS using this formula. From Figure 5(a), we see a minimum temperature T_0, and above this temperature there are two branches. The lower branch is always stable, while the Hawking-Page transition occurs at T_1 where the free energy vanishes.

For $Q \neq 0$, the phase structure resembles the Van der Waals transition, shown in Figure 4(b). The solid blue lines from top to bottom correspond to charges $Q = 0.11,\ 0.168678344129,\ 0.21$. Black holes with different charges exhibit different phase structures. For small charge, an unstable black hole interpolates between stable small and large holes. The small stable hole jumps to the large stable hole at critical temperature T_a. As charge increases to the critical value, the small and large holes merge, eliminating the unstable phase and creating an inflection point. The divergence of heat capacity in this case indicates a second-order phase transition. For charge exceeding the critical value, only one stable black hole exists at each temperature. The Van der Waals-like transition also appears in the F-T relation. From Figure 5(b), we observe a swallowtail structure corresponding to the unstable phase in the top curve of Figure 4(b). The critical temperature $T_a = 0.2825$ is clearly the horizontal coordinate of the junction between small and large black holes. Below T_a, the small black hole has lowest free energy and is stable; above T_a, the large black hole dominates. The junction' s non-smoothness indicates a first-order transition. From Figure 5(c), we see an inflection point corresponding to the middle curve in Figure 4(b), whose horizontal coordinate gives the second-order transition temperature T_c.

Similarly, we study the phase structure for $b = 4$ in the T-S plane. Using the same strategy as for $b = 5$, we find $Q_c = 0.16987452395$, $S_c = 0.502683752119928$, $T_c = 0.259166$. For $Q = 0$, the phase structure matches Figure 4(a) and will not be repeated.

For $Q \neq 0$, besides the Van der Waals-like transition, we find a novel phase transition for $Q = 0.115$ as observed in Figures 1 and 2, plotted in Figure 6(a). At the transition' s onset, an infinitesimally small black hole branch emerges.

This branch is unstable and short-lived, quickly becoming a small stable black hole. Additionally, we find two unstable regions and correspondingly two phase transition temperatures, labeled T_2 and T_3. In the F-T plane, we observe two swallowtail structures, shown in Figure 7(a). The horizontal coordinates of these swallowtails give the phase transition temperatures: besides the first-order transition temperature T_2 between small and large black holes, there is a new temperature T_3. However, this new transition cannot actually occur because above T_2, the large black hole has the lowest free energy and dominates spacetime, as seen in Figure 7(a). We therefore call this a pseudo phase transition. We also observe the Van der Waals-like transition, plotted in Figure 6(b). The solid blue lines from top to bottom correspond to charges $Q = 0.13$, 0.16987452395, 0.21. The first-order transition temperature T_a can be read from the swallowtail junction in Figure 7(b), and the second-order temperature T_c from the inflection point in Figure 7(c).

Following [?], we numerically verify Maxwell' s equal area law for first-order and pseudo phase transitions, which states:

$$\int_{S_{\min}}^{S_{\max}} T(S,Q)\,dS = T_x(S_{\max} - S_{\min}) \tag{2.10}$$

where $T(S,Q)$ is defined in Eq. (2.7), T_x is the phase transition temperature, and $S_{\min}$, $S_{\max}$ are the smallest and largest entropy values in the unstable region satisfying $T(S,Q) = T_x$. Typically this equation has three roots S_1, S_2, S_3, so $S_{\min} = S_1$, $S_{\max} = S_3$, and $T_x = T_a$. However, for $b = 4$, $Q = 0.115$, there are four roots. As seen in Figure 6(a), $S_{\min} = S_1$, $S_{\max} = S_3$ for the pseudo transition with $T_x = T_3$, while $S_{\min} = S_2$, $S_{\max} = S_4$ for the first-order transition with $T_x = T_2$. Table 1 lists the calculated results, showing $A_L = A_R$ within numerical accuracy for different b values, confirming the equal area law holds.

For the second-order transition, the heat capacity diverges near the critical point with critical exponent 2/3. As stated in [?], near the critical point there is always a linear relation:

$$\log|T - T_c| = 3\log|S - S_c| + \text{constant} \tag{2.11}$$

with slope 3. It is straightforward to show that for $b = 4, 5$ in our model, temperature and entropy satisfy this linear relation near the critical point. We will use Eq. (2.11) as a reference to check whether similar relations hold for second-order transitions in the two-point correlation function-temperature and entanglement entropy-temperature planes.

3. Phase Transition in the Framework of Holography

Having obtained the thermal entropy phase structure of Born-Infeld AdS black holes in the T-S plane, we now study the phase structure of two-point correlation functions and entanglement entropy in the field theory to determine if they exhibit similar phase structure and critical behavior.

3.1 Phase Structure Probed by Two-Point Correlation Function

According to AdS/CFT correspondence, in the large Δ limit, the equal-time two-point correlation function can be written as [**?**]:

$$\langle \mathcal{O}(t_0, x_i)\mathcal{O}(t_0, x_j)\rangle \sim e^{-\Delta L} \tag{3.1}$$

where Δ is the conformal dimension of the scalar operator $\mathcal{O}$ in the dual field theory, and L is the bulk geodesic length between points (t_0, x_i) and (t_0, x_j) on the AdS boundary. In our model, we choose $(\phi = \pi/2, \theta = 0)$ and $(\phi = \pi/2, \theta = \theta_0)$ as the two boundary points. Parameterizing the trajectory with θ, the proper length is:

$$L = 2\int_0^{\theta_0} \sqrt{\frac{r'(\theta)^2}{f(r)} + r^2}\, d\theta \tag{3.2}$$

where $r' = dr/d\theta$. Treating θ as time and using the Euler-Lagrange equation yields the equation of motion for $r(\theta)$. With boundary conditions:

$$r'(0) = 0, \quad r(0) = r_0 \tag{3.3}$$

we obtain numerical solutions for $r(\theta)$. To explore whether boundary region size affects phase structure, we choose $\theta_0 = 0.2, 0.3$ as examples. For fixed θ_0, the geodesic length diverges and must be regularized by subtracting the geodesic length in pure AdS with the same boundary region, denoted L_0. This requires a UV cutoff for each case, chosen as $r(0.199)$ and $r(0.299)$ respectively. The regularized geodesic length is labeled δL. During numerics, we set the AdS radius $l = 1$.

We plot the T-δL relation for $b = 5$ in Figures 8 and 9. The solid blue line in Figure 8(a) corresponds to $Q = 0$, while dashed blue lines from top to bottom in Figure 8(b) correspond to $Q = 0.11$, 0.168678344129, 0.21. These figures show that δL exhibits phase structure similar to thermal entropy in Figure 4. The two-point correlation function probes both Hawking-Page and Van der Waals-like phase transitions. Notably, the minimum temperature T_0, Hawking-Page temperature T_1 in Figure 8(a), first-order temperature T_a, and second-order temperature T_c in Figure 8(b) exactly match those in the T-S plane. This conclusion remains unaffected as θ_0 varies within reasonable bounds, as seen in Figures 8 and 9.

The two-point correlation function also probes the phase structure for $b = 4$, shown in Figures 10 and 11. The solid blue line in Figure 10(a) corresponds to $Q = 0.115$, while solid blue lines from top to bottom in Figure 10(b) correspond to $Q = 0.13$, 0.16987452395, 0.21. As with thermal entropy in the T-S plane, we find a novel phase structure besides the Van der Waals-like transition in the δL-T plane, with two unstable regions and correspondingly two phase transition temperatures T_2 and T_3. To precisely locate T_2, T_3, and T_a, we examine the equal area law for first-order and pseudo transitions. To locate T_c, we obtain the critical exponent 2/3 for the second-order transition.

In the δL-T plane, we define the equal area law as:

$$\int_{\delta L_{\min}}^{\delta L_{\max}} T(\delta L)\, d\delta L = T_x(\delta L_{\max} - \delta L_{\min}) \tag{3.4}$$

where $T(\delta L)$ is an interpolating function from numerical results, T_x is the phase transition temperature, and $\delta L_{\min}$, $\delta L_{\max}$ are the smallest and largest values in the unstable region satisfying $T(\delta L) = T_x$. Similar to the T-S plane, for $b = 5$, $Q = 0.11$ and $b = 4$, $Q = 0.13$, $T_x = T_a$, while for $b = 4$, $Q = 0.115$, $T_x = T_2$ for the first-order transition and $T_x = T_3$ for the pseudo transition. Table 2 lists the results for different b and θ_0, showing $A_L = A_R$ within reasonable accuracy. Thus, the equal area law holds in the δL-T plane, independent of the Born-Infeld parameter and boundary region size.

To obtain the critical exponent for the second-order transition in the δL-T plane, we first define an analogous heat capacity:

$$C = T\left(\frac{\partial S}{\partial T}\right) \tag{3.5}$$

If a relation similar to Eq. (2.11) is satisfied, the critical exponent follows immediately. We therefore examine the logarithmic relation:

$$\log|T - T_c| = 3\log|\delta L - \delta L_c| + \text{constant}$$

where T_c is the previously mentioned critical temperature and δL_c is obtained numerically from $T(\delta L) = T_c$. Figure 12 plots $\log|T - T_c|$ versus $\log|\delta L - \delta L_c|$ for different θ_0 and b, with fitted straight lines:

$$\begin{cases} \log|T-T_c| = 25.0567 + 3.0113\log|\delta L - \delta L_c| & \text{for } b=5, \theta_0 = 0.2 \\ \log|T-T_c| = 21.207 + 3.0402\log|\delta L - \delta L_c| & \text{for } b=5, \theta_0 = 0.3 \\ \log|T-T_c| = 23.8282 + 3.01531\log|\delta L - \delta L_c| & \text{for } b=4, \theta_0 = 0.2 \\ \log|T-T_c| = 20.3704 + 3.03817\log|\delta L - \delta L_c| & \text{for } b=4, \theta_0 = 0.3 \end{cases} \tag{3.6}$$

The slope is approximately 3, indicating the critical exponent is nearly 2/3 for the analogous heat capacity, confirming a second-order transition at temperature T_c.

3.2 Phase Structure Probed by Holographic Entanglement Entropy

According to the holographic formula [?, ?], entanglement entropy is given by the area A_Σ of a minimal surface Σ anchored on the boundary entangling surface $\partial\Sigma$:

$$S_A = \frac{A_\Sigma}{4G} \tag{3.7}$$

where G is Newton' s constant. We take region Σ to be a spherical cap on the boundary delimited by $\theta = \theta_0$ (to avoid contamination by thermal entropy, we choose a small entangling region as in [?, ?]). Based on the area definition and Eq. (2.2), Eq. (3.7) becomes:

$$A_\Sigma = 2\pi \int_0^{\theta_0} r\sin\theta \sqrt{\frac{r'^2}{f(r)} + r^2}\, d\theta \tag{3.8}$$

where $r' = dr/d\theta$. Using the Euler-Lagrange equation gives the equation of motion for $r(\theta)$. With boundary conditions from Eq. (3.3), we obtain numerical solutions for $r(\theta)$. The entanglement entropy diverges at the boundary and must be regularized by subtracting the pure AdS entanglement entropy with the same entangling surface. We label the regularized entanglement entropy as δS, choose boundary region size $\theta_0 = 0.2$, and set the UV cutoff in the dual field theory to $r(0.199)$.

Isocharge curves for $b = 5$ and $b = 4$ in the entanglement entropy-temperature plane are plotted in Figures 13 and 14. As with the two-point correlation function, for $b = 5$ we consider charges $Q = 0$, 0.11, 0.168678344129, 0.21, and for $b = 4$ we choose $Q = 0.115$, 0.13, 0.16987452395, 0.21. Figure 13 shows that entanglement entropy exhibits both Hawking-Page and Van der Waals-like phase transitions as charge increases from 0 to 0.21. Entanglement entropy also shows the novel phase structure seen in thermal entropy and two-point correlation function, displayed in Figure 14(a).

We use Maxwell' s equal area law to locate the first-order transition temperature T_a in Figures 13(b) and 14(b), T_2 in Figure 14(a), and pseudo transition temperature T_3. In the δS-T plane, the equal area law is:

$$\int_{\delta S_{\min}}^{\delta S_{\max}} T(\delta S)\, d\delta S = T_x(\delta S_{\max} - \delta S_{\min}) \tag{3.9}$$

where $T(\delta S)$ is an interpolating function from numerical results, and $\delta S_{\min}$, $\delta S_{\max}$ are the smallest and largest roots in unstable regions satisfying $T(\delta S) = T_x$. The phase transition temperature T_x depends on Q and b. Table 3 lists $\delta S_{\min}$, $\delta S_{\max}$, and A_L, A_R for different Q and b, showing $A_L \approx A_R$ for each case. Thus, the equal area law holds for both first-order and pseudo transitions in the entanglement entropy-temperature plane, consistent with thermal entropy and two-point correlation function results.

To confirm T_c as the second-order transition temperature in the δS-T plane, we check whether it satisfies a relation similar to Eq. (2.11). Figure 15 plots $\log|T - T_c|$ versus $\log|\delta S - \delta S_c|$ for different b and Q, where δS_c is the critical entropy obtained numerically from $T(\delta S) = T_c$. The fitted lines are:

$$\begin{cases} \log|T - T_c| = 17.3605 + 2.91954 \log|\delta S - \delta S_c| & \text{for } b = 5, \theta_0 = 0.2 \\ \log|T - T_c| = 17.2131 + 2.93553 \log|\delta S - \delta S_c| & \text{for } b = 4, \theta_0 = 0.2 \end{cases} \tag{3.10}$$

For fixed b and Q, the slope is always approximately 3, consistent with thermal entropy results. Thus, entanglement entropy also exhibits a second-order phase transition at temperature T_c, matching thermal entropy and two-point correlation function behavior.

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4. Concluding Remarks

Investigating black hole phase structure is crucial for understanding whether and how black holes emerge. Typically, this is achieved by studying thermodynamic quantity relations in a fixed ensemble. In this paper, we demonstrated that black hole phase structure can also be probed by two-point correlation functions and holographic entanglement entropy, providing a new holographic strategy to understand black hole phase structure.

Specifically, we first investigated Born-Infeld-anti-de Sitter black hole phase structure in the T-S plane for a fixed charge ensemble. We found that phase structure depends not only on Q or b individually, but also on the combination bQ. For $b = 5$ (satisfying $bQ > 0.5$), the black hole resembles Reissner-Nordström-AdS black holes and undergoes Van der Waals-like phase transitions. For $b = 4$ (with $bQ < 0.5$), besides the Van der Waals-like transition, we observed a novel phase structure. Using two-point correlation functions and holographic entanglement entropy, we further probed the phase structure and found both probes exhibited the same phase structure as thermal entropy for both $bQ > 0.5$ and $bQ < 0.5$, regardless of boundary region size. This conclusion was reinforced by verifying the equal area law for first-order and pseudo transitions and calculating the critical exponent of analogous heat capacity for second-order transitions.

Previous investigations of Born-Infeld-anti-de Sitter black holes focused mainly on $bQ > 0.5$ [**?**, **?**, **?**, **?**]. We found the $bQ < 0.5$ case also interesting. As bQ approaches 0.5, a new extremal small black hole branch emerges compared to Reissner-Nordström-AdS black holes, creating two unstable regions and two phase transition temperatures. The high-temperature transition is pseudo because spacetime is dominated by the large black hole, as observed in the F-T relation.

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