Entanglement entropy in a holographic p-wave superconductor model (postprint)

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Abstract

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Full Text

Preamble

Entanglement Entropy in a Holographic p-Wave Superconductor Model

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Abstract

In a recent paper [1309.4877], a holographic p-wave model was proposed in an Einstein-Maxwell-complex vector field theory with a negative cosmological constant. The model exhibits a rich phase structure depending on the mass and charge of the vector field. We investigate the behavior of the entanglement entropy of the dual field theory in this model. When these two model parameters change, we observe second-order, first-order, and zeroth-order phase transitions from the behavior of the entanglement entropy at intermediate temperatures. These observations imply that entanglement entropy can indicate not only the occurrence of phase transitions but also their order. The entanglement entropy is indeed a good probe of phase transitions. Furthermore, the "retrograde condensation," which is a subdominant phase, is also reflected in the entanglement entropy.

Introduction

Over the past years, the AdS/CFT correspondence [Maldacena:1997re, Gubser:1998bc, Witten:1998qj] has been extensively applied to the study of superconductors and superfluids in condensed matter physics (for reviews, see [Hartnoll:2009sz, Herzog:2009xv, McGreevy:2009xe, Horowitz:2010gk, Cai:2015cya]). Recently, we proposed a holographic model of p-wave superconductors in [Cai:2013pda, Cai:2013aca] by introducing a complex vector field ρ_μ charged under a Maxwell gauge field A_μ in the AdS bulk. This model contains two parameters: the mass and charge of the vector field. Depending on these model parameters, it has been shown that the model exhibits a rich phase structure [Cai:2013aca, Cai:2014znn]. In this model, second-order, first-order, and even zeroth-order phase transitions may occur. We also found the so-called "retrograde condensation," though it is subdominant in the sense that its free energy is much larger than that of the black hole solution without vector hair. Compared with the holographic p-wave model proposed in [Gubser:2008wv], which introduces an SU(2) gauge field in the bulk, the new p-wave model is a generalization in that the vector field has a general mass and gyromagnetic ratio [Cai:2013pda].

Entanglement entropy is a good probe of quantum phase transitions and various phases in quantum field theory and many-body systems. The holographic entanglement entropy proposed in [Ryu:2006bv] provides an effective way to calculate the entanglement entropy for strongly coupled field theories (for reviews, see [Nishioka:2009un, Takayanagi:2012kg]). The authors of [Albash:2012pd, Cai:2012nm, Arias:2012py, Peng:2014ira, Yao:2014fua] studied the behavior of entanglement entropy in holographic conductor/superconductor phase transitions, including both s-wave and p-wave cases. The results showed that entanglement entropy decreases as temperature is lowered, indicating that degrees of freedom are reduced at lower temperatures. Moreover, the behavior of entanglement entropy changes dramatically when the order of the phase transition changes, which implies that entanglement entropy is indeed a good probe of phase transitions. The behavior of entanglement entropy for holographic s-wave and p-wave superconductor/insulator models at zero temperature has also been investigated in [Cai:2012sk, Cai:2012es, Yao:2014kch]. This case is very similar to the study of confinement/deconfinement phase transitions using holographic entanglement entropy [Nishioka:2006gr, Klebanov:2007ws]. Unlike the conductor/superconductor phase transition, the entanglement entropy in the s-wave superconductor/insulator case [Cai:2012sk, Cai:2012es, Yao:2014kch] as a function of chemical potential is not monotonic: at the beginning of the transition, the entropy first increases and reaches its maximum at some chemical potential, then decreases monotonically.

In this paper, we study the entanglement entropy in this new holographic p-wave superconductor model [Cai:2013pda, Cai:2013aca]. The entanglement entropy is calculated for a straight strip geometry using the holographic proposal. We find that the behavior of entanglement entropy changes qualitatively in different phases as we alter the mass squared m^2 and charge q of the vector field. Our results show that entanglement entropy can capture the characteristics of phase transitions and various phases in the holographic superconductor model. By comparing the entanglement entropy with the thermal entropy of the bulk black holes during the entire phase transition process, we see the possibility of understanding black hole entropy as entanglement entropy [Jacobson:1994iw, Kabat:1995jq, Solodukhin:2006xv, Emparan:2006ni]. Furthermore, the "retrograde condensation," which is a subdominant phase, is also reflected in the entanglement entropy.

This paper is organized as follows. In the next section, we briefly review the holographic p-wave superconductor model. The full system with backreaction is solved using the shooting method, and the main phase structure of the model is summarized. Section 3 is devoted to exploring the behavior of entanglement entropy in the p-wave superconductor model. We present numerical results in this section, and for each given m^2, we scan a wide range of q to find all possible behaviors of entanglement entropy. Conclusions and discussions are included in Section 4.

2 The Holographic Model

We begin with the holographic model of p-wave superconductors proposed in [Cai:2013pda, Cai:2013aca]. The action is given by

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\rho^\dagger_{\mu\nu}\rho^{\mu\nu} - m^2\rho^\dagger_\mu\rho^\mu\right]$$

where $\kappa^2 = 8\pi G$ with G being the gravitational constant, and L is the AdS spacetime radius. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Maxwell field strength, and ρ_μ is the complex vector field. The field strength for the vector field is $\rho_{\mu\nu} = D_\mu\rho_\nu - D_\nu\rho_\mu$ with the covariant derivative $D_\mu = \nabla_\mu - iqA_\mu$. The parameters m^2 and q are the mass squared and charge of the complex vector field, respectively.

Varying the action yields the equations of motion for the gauge field A_μ and charged vector field ρ_μ:

$$\nabla^\nu F_{\nu\mu} = iq(\rho^\nu \rho^\dagger_{\nu\mu} - \rho^{\nu\dagger}\rho_{\nu\mu})$$

$$D^\nu \rho_{\nu\mu} - m^2 \rho_\mu = 0$$

and the corresponding Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{3}{L^2}g_{\mu\nu} = \frac{\kappa^2}{2}\left[F_{\mu\lambda}F_\nu{}^\lambda + \rho^\dagger_{\mu\lambda}\rho_\nu{}^\lambda + m^2\rho^\dagger_\mu\rho_\nu\right] - \frac{\kappa^2}{4}g_{\mu\nu}\left[F_{\lambda\sigma}F^{\lambda\sigma} + \rho^\dagger_{\lambda\sigma}\rho^{\lambda\sigma} + 2m^2\rho^\dagger_\lambda\rho^\lambda\right]$$

The system admits an analytical solution with vanishing ρ_μ, corresponding to the normal phase, which is simply the planar AdS Reissner-Nordström black hole [Cai:1996eg]:

$$ds^2 = \frac{r^2}{L^2}\left[-A(r)dt^2 + dx^2 + dy^2\right] + \frac{L^2}{r^2 A(r)}dr^2$$

with

$$A(r) = 1 - \frac{r_h^3}{r^3} + \frac{\mu^2}{4r^2}\left(1 - \frac{r_h}{r}\right)$$

and

$$\phi(r) = \mu\left(1 - \frac{r_h}{r}\right)$$

where r_h is the horizon radius and μ is the chemical potential of the dual field theory. The Hawking temperature of the black hole is

$$T = \frac{r_h}{4\pi}\left(3 - \frac{\mu^2}{4r_h^2}\right)$$

When we tune the temperature, the system exhibits an instability that triggers condensation of the charged vector field ρ_μ. Here we consider the following ansatz [Cai:2013aca]:

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2 h(r)dx^2 + r^2 dy^2$$

$$\rho_\nu dx^\nu = \rho_x(r)dx, \quad A_\nu dx^\nu = \phi(r)dt$$

The temperature of the black hole is given by

$$T = \frac{f'(r_h)e^{-\chi(r_h)/2}}{4\pi}$$

and the thermal entropy is given by the Bekenstein-Hawking area formula:

$$S = \frac{2\pi V_2 r_h^2}{\kappa^2}$$

where V_2 is the area spanned by coordinates x and y. According to the AdS/CFT correspondence, the solution with vanishing ρ_ν corresponds to a conductor (normal) phase, while the solution with nontrivial vector hair ρ_ν corresponds to the superconducting phase of the dual field theory. The transition from a black hole without hair to one with nontrivial vector hair mimics the conductor/superconductor phase transition. To study the behavior of entanglement entropy during this process, we must first obtain the hairy black hole solution.

With the ansatz above, the independent equations of motion become:

$$\phi'' + \left(\frac{2}{r} - \frac{\chi'}{2}\right)\phi' - \frac{2q^2\rho_x^2}{r^2 f}\phi = 0$$

$$\rho_x'' + \left(\frac{2}{r} + \frac{f'}{f} - \frac{\chi'}{2}\right)\rho_x' + \left(\frac{e^\chi q^2\phi^2}{f^2} - \frac{m^2}{f}\right)\rho_x = 0$$

$$\chi' + \frac{2e^\chi q^2\phi^2\rho_x^2}{r^3 f^2} = 0$$

$$h'' + \left(\frac{4}{r} - \frac{\chi'}{2}\right)h' - \frac{2e^\chi q^2\phi^2\rho_x^2}{r^4 f h} = 0$$

$$f' + \left(\frac{1}{r} - \frac{h'}{2h}\right)f + \frac{re^\chi\phi'^2}{4} + \frac{m^2\rho_x^2}{2r} - \frac{3r}{L^2} = 0$$

where the prime denotes derivative with respect to r. We use the shooting method to solve these equations. First, we must specify the boundary conditions for the system. Near the AdS boundary $r \to \infty$, the fields have the following expansions:

$$f = r^2\left(1 + \frac{f_3}{r^3} + \cdots\right), \quad \chi = \frac{\chi_3}{r^3} + \cdots$$

$$\phi = \mu - \frac{\rho}{r} + \cdots, \quad h = 1 + \frac{h_3}{r^3} + \cdots$$

$$\rho_x = \frac{\rho_{x-}}{r^{\Delta_-}} + \frac{\rho_{x+}}{r^{\Delta_+}} + \cdots, \quad \Delta_\pm = \frac{3 \pm \sqrt{9 + 4m^2}}{2}$$

According to the AdS/CFT dictionary, the constants μ and ρ can be interpreted as the chemical potential and charge density in the dual field theory, ρ_{x-} is the source of the dual operator, and ρ_{x+} gives its expectation value. To require spontaneous breaking of the U(1) symmetry, we set $\rho_{x-} = 0$ in our numerical calculations.

We impose regularity conditions at the horizon $r = r_h$. Specifically, we have $f(r_h) = 0$, $\phi(r_h) = 0$, and finite values for $\rho_x(r_h)$, $\phi'(r_h)$, $h(r_h)$, and $\chi(r_h)$. This leaves us with five independent parameters. The scaling symmetry $(t, x, y) \to \lambda(t, x, y)$, $r \to \lambda^{-1} r$, $(\phi, \rho_x) \to \lambda(\phi, \rho_x)$, $f \to \lambda^2 f$, $e^\chi \to \lambda^2 e^\chi$ can be used to set $r_h = 1$ for numerical convenience. Similarly, the scaling symmetries allow us to set $\chi(r_h) = 0$ and $h(r_h) = 1$. Thus we finally have two independent parameters: $\rho_x(r_h)$ and $\phi'(r_h)$. Given $\phi'(r_h)$ as the shooting parameter to match the source-free condition $\rho_{x-} = 0$, we can solve the fully coupled system of differential equations.

To determine which phase is thermodynamically favored, we calculate the free energy of the system for both the normal and condensed phases. Working in the grand canonical ensemble with fixed chemical potential, the free energy Ω can be expressed as

$$\Omega = \frac{V_2}{\kappa^2}(f_3 + h_3 - \chi_3)$$

For the normal phase given above, we have $f_3 = -\mu^2/4$, and $h_3 = \chi_3 = 0$.

We have studied the phase structure of the model by investigating the thermodynamics of various solutions. There exists a critical mass squared $m_c^2 = 0$. The system shows qualitatively different properties for the cases $m^2 > m_c^2$ and $m^2 < m_c^2$.

When $m^2 > m_c^2$, there exists a critical charge q_c that depends on the value of m^2. If $q > q_c$, the system undergoes a second-order phase transition at critical temperature $T = T_c$ from the conductor (normal) phase for $T > T_c$ to the superconducting phase for $T < T_c$. On the other hand, if $q < q_c$, the phase transition becomes first-order, as shown in Figure 1.

When $m^2 < m_c^2$, the range of charge q is divided into three regimes by two critical charges, q_α and q_β, which also depend on m^2. If $q > q_\alpha$, the system is in a normal phase when $T > T_{2c}$, in a superconducting phase when $T_{0c} < T < T_{2c}$, and in the normal phase when $T < T_{0c}$. There exists a second-order phase transition at

$T = T_{2c}$, where the free energy is continuous, and a zeroth-order phase transition at $T = T_{0c}$, where the free energy is discontinuous. If $q_\beta < q < q_\alpha$, the second-order phase transition is replaced by a first-order one at $T = T_{1c}$. If $q < q_\beta$, the black hole solution with vector hair can exist, but its free energy is always larger than that without vector hair. This condensation is called "retrograde condensation." The solution is subdominant and therefore does not appear in the phase diagram. In this case, the system is always in the normal phase. These features are clearly visible in Figure 2.

[Figure 1: see original paper] shows the grand potential Ω as a function of temperature for $m^2 = 3/4$ with $q = 3/2$ (left plot) and $q = 6/5$ (right plot). The critical temperature T_c marks the onset of the superconducting phase. The physical curve is obtained by choosing the lowest grand potential at fixed T. The $q = 3/2$ plot shows a typical second-order phase transition, while the $q = 6/5$ plot displays a first-order phase transition.

In summary, depending on the model parameters m^2 and q, the model exhibits a rich phase structure, detailed in Table 1. In the spirit of AdS/CFT correspondence, the mass squared m^2 of the vector field is related to the conformal dimension of the dual vector operator, while the charge q corresponds to the "charge" of the condensate. From the bulk perspective, q is the coupling parameter between the vector field and the bulk Maxwell field and governs the backreaction strength of matter fields on the background geometry. In this sense, q plays the same role as the coupling parameter $\hat{g}^2$ in Einstein-Yang-Mills theory [Ammon:2009fe]. Thus one may think that q counts additional degrees of freedom in the dual CFT. In this way, the mass squared and charge of the vector field control the phase structure of the dual CFT.

We note that a zeroth-order phase transition with discontinuous free energy appears strange at first glance, but it has been argued that such transitions could appear in superconductivity and superfluidity [Maslov:2004]. In that reference, an exactly solvable model was also proposed. Furthermore, zeroth-order phase transitions appear in the extended phase space of AdS black hole thermodynamics [Altamirano:2013uqa, Gunasekaran:2012dq]. However, we remind the reader that the zeroth-order phase transition might be unphysical in this holographic superconductor model because we have only considered two phases: the normal phase described by the RN-AdS black hole and the superconducting phase described by the black hole with vector hair, both of which preserve translation symmetry in the transverse directions. If we consider other components of the complex vector field and/or cases without transverse translation symmetry, we might find other phases with lower free energy that are more stable, and the zeroth-order phase transition discussed above would be replaced by first- or second-order transitions. In this paper, however, we limit ourselves to the case summarized in Table 1.

[Figure 2: see original paper] shows the free energy Ω as a function of temperature with vector condensation (solid red) and without vector condensation (dashed blue) for $m^2 = 3/16$ and various values of q. The $q = 2$ plot shows a second-order phase transition, but the condensed phase terminates at a finite low temperature T_0. The $q = 39/40$ plot exhibits a first-order phase transition, but the superconducting phase also terminates at finite low temperature T_0. The $q = 19/20$ and $q = 9/10$ plots demonstrate that the condensed phase has free energy much larger than the normal phase and is therefore not thermodynamically favored.

3 Entanglement Entropy

Within the AdS/CFT correspondence, a holographic method to calculate entanglement entropy was proposed in [Ryu:2006bv]. For a conformal field theory with a dual gravitational configuration living in one higher dimension, the entanglement entropy of a subsystem and its complement can be obtained by finding the minimal area surface γ extending into the bulk with the same boundary $\partial\gamma = \partial\mathcal{A}$. The entanglement entropy is given by the "area law":

$$S_{\mathcal{A}} = \frac{2\pi}{\kappa^2}\text{Area}(\gamma)$$

In this section, we study the behavior of entanglement entropy in the holographic p-wave superconductor model. We consider a belt geometry with finite width ℓ along the x direction and infinite extent in the y direction. The holographic dual surface $\gamma_{\mathcal{A}}$ is defined as a two-dimensional surface at $t = 0$, $r = r(x)$, with $-\ell/2 < x < \ell/2$ and $0 < y < R$, where R is the regularized length in the y direction. To avoid UV divergence, we introduce a UV cutoff at $r = 1/\epsilon$. Specifically, the holographic surface $\gamma_{\mathcal{A}}$ starts from $x = -\ell/2$ at the boundary $r = 1/\epsilon$, extends into the bulk until it reaches $r = r_*$ at $x = 0$, then returns symmetrically to the boundary at $x = +\ell/2$. The smooth configuration is shown in Figure 3. Note that a piecewise configuration might also exist where the dual surface goes straight down from the boundary to the horizon (see, e.g., Figure 1 in [Bah:2007ah]). However, it has been suggested that generically the smooth configuration dominates at finite temperature involving black hole horizons [Bah:2007ah]. Therefore, following previous studies [Albash:2012pd, Cai:2012nm, Arias:2012py], we consider only the smooth configuration.

[Figure 3: see original paper] shows the minimal surface $\gamma_{\mathcal{A}}$ corresponding to the strip region. The quantity ℓ sets the size of the region on the boundary.

The entanglement entropy for such a subsystem is given by

$$S_E = \frac{2\pi R}{\kappa^2}\int_{-\ell/2}^{\ell/2} dx\, r^2 h(r)\sqrt{\frac{1}{f(r)} + \frac{r'^2}{r^4 h(r)^2}}$$

where $r' = dr/dx$. The width ℓ and the turning point r_* are related by

$$\ell = 2\int_{r_*}^{1/\epsilon} dr\, \frac{r^2 h(r)}{r_*^2 h(r_*)}\sqrt{\frac{f(r)}{r^4 h(r)^2 - r_*^4 h(r_*)^2}}$$

We are interested in the finite part S_E of the entanglement entropy, which has physical relevance. Using the background solutions $h(r)$ and $f(r)$ obtained by solving the equations of motion, we can calculate the entanglement entropy numerically.

Our results show that the entanglement entropy S_E as a function of strip width ℓ behaves similarly for different parameters m^2 and q. The entanglement entropy versus strip width is shown in Figure 4 for $m^2 = 3/4$ and $q = 3/2$. In this case, the critical temperature is $T_c \approx 0.0179$ [Cai:2013aca]. Different colored curves show how the entanglement entropy changes with belt width at various temperatures. Note that we work in the grand canonical ensemble with fixed chemical potential $\mu = 1$. From top to bottom, the temperature decreases. The topmost curve corresponds to the critical temperature T_c, which is identical to the normal phase case. Figure 4 shows that the slope of the curves decreases as temperature is lowered in the superconducting phase, as expected because lower temperatures lead to more condensation of degrees of freedom.

For a fixed temperature, say $T = 0.0165$, the blue curve describes a monotonically increasing entanglement entropy with respect to belt width. The entanglement entropy is dominated by the connected surface $\gamma_{\mathcal{A}}$. Wider belt widths correspond to larger areas of the holographic surface. This dependence of entanglement entropy on ℓ is nontrivial. Specifically, for large ℓ, S_E changes linearly with ℓ because in this limit the main contribution to the integrals comes from the region near $r \approx r_h$. Near the horizon, we can deduce the linear relation $S_E \sim \ell$. In contrast, S_E appears to diverge as a power law when ℓ approaches zero.

As shown in Table 1, the phase structure differs qualitatively between the $m^2 > m_c^2$ and $m^2 < m_c^2$ cases. Below we explore the behavior of entanglement entropy quantitatively for these two cases.

3.1 The Case with $m^2 > m_c^2$

We take $m^2 = 3/4$ as a typical example. For other mass values, the qualitative behavior of entanglement entropy is the same. In this case, the critical charge is $q_c \approx 1.3575$ [Cai:2013aca].

[Figure 5: see original paper] shows that the entanglement entropy is continuous at the critical temperature T_c, but its slope is not. This discontinuity signals a significant reorganization of the system's degrees of freedom. Since a condensate is generated at the transition point, we expect a reduction in degrees of freedom. The behavior of entanglement entropy versus temperature indicates a typical second-order phase transition.

When we decrease q to $q = 6/5$, which is less than $q_c \approx 1.3575$, the entanglement entropy as a function of temperature is presented in Figure 6. In this case, the entanglement entropy jumps from the normal phase to the superconducting phase at the critical temperature, showing first-order phase transition behavior, consistent with the thermal entropy behavior. This example demonstrates that entanglement entropy can indeed reveal both the occurrence and order of phase transitions.

From Figures 5 and 6, we also see that the qualitative behavior of entanglement entropy is the same for belt widths $\ell = 4$ and $\ell = 6$. The only difference is that the entanglement entropy value is larger for larger ℓ at the same temperature T, consistent with the observation in Figure 4.

Note that the behavior of entanglement entropy observed here is qualitatively the same as that of the thermal entropy considered in [Cai:2013aca]. Since thermal entropy is equivalent to black hole entropy in the holographic setup, our results suggest that black hole entropy could be understood as entanglement entropy [Jacobson:1994iw, Kabat:1995jq, Solodukhin:2006xv, Emparan:2006ni].

3.2 The Case with $m^2 < m_c^2$

We take $m^2 = 3/16$ as a typical example (the Breitenlohner-Freedman bound is still satisfied). In this case, the critical charges are $q_\alpha \approx 1.0175$ and $q_\beta \approx 0.9537$ [Cai:2013aca]. We find different behaviors of entanglement entropy for the cases $q > q_\alpha$, $q_\alpha > q > q_\beta$, and $q < q_\beta$. Figures 7-10 plot entanglement entropy versus temperature for $q = 2$, $q = 39/40$, $q = 19/20$, and $q = 9/10$, respectively.

[Figure 7: see original paper] shows entanglement entropy versus temperature for $q = 2$ as an example of the $q > q_\alpha$ case. At the second-order transition point $T = T_{2c}$, the entanglement entropy is continuous but its slope is not, while at the zeroth-order transition point $T = T_{0c}$, the entanglement entropy has a jump. Additionally, in the superconducting phase, the entanglement entropy has two branches: the solid green curve is physically favored, while the dashed curve is not physically favored since the corresponding solution has higher free energy.

For the case $q_\beta < q < q_\alpha$, we consider the example with $q = 39/40$. The behavior of entanglement entropy as a function of temperature is shown in Figure 8. The entanglement entropy has a jump both at the first-order transition point $T = T_{1c}$ and at the zeroth-order transition point $T = T_{0c}$, but the two situations differ. For the former, the entanglement entropies in the normal and superconducting phases can be connected by a metastable superconducting phase (dashed curve), while for the latter, no such metastable phase exists.

For the case $q < q_\beta$, we consider two examples: $q = 19/20$ and $q = 9/10$. Their entanglement entropy behaviors are shown in Figures 9 and 10, respectively. In this case, condensation appears in the high-temperature regime, and the hairy black hole solutions have higher free energy than the RN-AdS black hole at the same temperature; therefore, the hairy black hole solutions are not physically favored (bottom left and bottom right plots in Figure 2). We see that the entanglement entropy S_E is multi-valued for $q = 19/20$, while it decreases monotonically with increasing temperature for $q = 9/10$ for the hairy black hole solutions.

Once again, we find that the qualitative behaviors of entanglement entropy are the same for different belt widths. Comparing the entanglement entropy behaviors with the corresponding thermal entropy behaviors studied in [Cai:2013aca], we find they are qualitatively identical. This further supports that entanglement entropy is a good measure of degrees of freedom for quantum field theories and demonstrates that it is a good probe of phase transitions and various phases in holographic superconductor models.

4 Conclusions and Discussions

In recent works [Cai:2013pda, Cai:2013aca, Cai:2014znn], we constructed a holographic p-wave superconductor model in four-dimensional Einstein-Maxwell-complex vector field theory with a negative cosmological constant. Depending on the mass squared m^2 and charge q of the complex vector field ρ_μ, the model exhibits a rich phase structure. Second-order, first-order, and even zeroth-order phase transitions can appear, and the so-called "retrograde condensation" can occur in some parameter regime. In this paper, we continued studying this model by investigating the behavior of holographic entanglement entropy for a belt geometry at the AdS boundary.

The main results are presented in Figures 4-10. Figure 4 shows the behavior of entanglement entropy with respect to belt width at different temperatures in the superconducting phase. For fixed temperature, entanglement entropy increases with belt width, while for fixed width, it decreases as temperature is lowered. This is expected because entanglement entropy measures degrees of freedom in the dual field theory, and more degrees of freedom condense at lower temperatures.

Figures 5-6 and 7-10 show the behavior of entanglement entropy with respect to temperature in various phases, depending on model parameters m^2 and q. We observed that at second-order phase transition points, entanglement entropy is continuous but its slope is discontinuous, while at first- and zeroth-order transition points, entanglement entropy has a jump. However, the latter two cases differ: for first-order transitions, the entanglement entropies of the normal and superconducting phases can be connected by a metastable condensed phase, while for zeroth-order transitions, no such metastable phase exists. We reiterate that the zeroth-order phase transition discussed here might be unphysical, as mentioned above, because more stable nontrivial black hole solutions might exist in this model.

In this study, we found that entanglement entropy behavior always shares similarities with thermal entropy [Cai:2013aca]. This similarity can be understood because entanglement entropy is contaminated by thermal fluctuations at finite temperature and approaches thermal entropy at high temperatures. In this sense, entanglement entropy is a unique order parameter that can be used even at zero temperature, and it deserves further investigation in holographic superconductor/insulator transitions in the Einstein-Maxwell-complex vector model. Indeed, entanglement entropy in some superconductor/insulator models at zero temperature has been studied, revealing interesting behaviors [Cai:2012sk, Cai:2012es, Yao:2014kch].

Note that we have only studied entanglement entropy with finite width along the x direction. In this model, spatial rotational symmetry is broken in the superconducting phase. To capture anisotropy in the x and y directions, one should also study entanglement entropy along the y direction. We do not calculate it here because the behavior is expected to be qualitatively similar to the x direction case. As a nonlocal quantity, entanglement entropy describes new degrees of freedom emerging in the superconducting phase. Since the degree of anisotropy is characterized by the condensate value, no qualitative difference is expected between entanglement entropy along x and y directions [Cai:2013aqa].

Furthermore, besides entanglement entropy, another nonlocal quantity—the Wilson loop—can describe aspects of gauge field theory. Studying this quantity can also shed light on holographic superconducting phase transitions. We hope to report further progress on these issues in the future.

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TABLE:1 The phase transition and its order with respect to the charge q and mass squared m^2. Here N stands for conductor (normal) phase with RN-AdS black hole solution, while SC denotes the superconducting phase with black hole solutions with vector hair. 0th, 1st, and 2nd indicate zeroth-order, first-order, and second-order phase transitions, respectively. $m_c^2 = 0$. q_c, q_α, q_β are critical values depending on m^2. T_c, T_{0c}, T_{1c}, and T_{2c} are critical temperatures depending on m^2 and q.

Charge q	Phase transition and order for $m^2 > m_c^2$	Phase transition and order for $m^2 < m_c^2$
$q > q_c$ (for $m^2 > m_c^2$) or $q > q_\alpha$ (for $m^2 < m_c^2$)	$T > T_c$: N; $T = T_c$: 2nd; $T < T_c$: SC	$T > T_{2c}$: N; $T = T_{2c}$: 2nd; $T_{0c} < T < T_{2c}$: SC; $T = T_{0c}$: 0th; $T < T_{0c}$: N
$q < q_c$ (for $m^2 > m_c^2$) or $q_\beta < q < q_\alpha$ (for $m^2 < m_c^2$)	$T > T_c$: N; $T = T_c$: 1st; $T < T_c$: SC	$T > T_{1c}$: N; $T = T_{1c}$: 1st; $T_{0c} < T < T_{1c}$: SC; $T = T_{0c}$: 0th; $T < T_{0c}$: N
$q < q_\beta$ (for $m^2 < m_c^2$)	-	No phase transition (retrograde condensation)

Note: Figure translations are in progress. See original paper for figures.