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Full Text

Preamble

Coupled-Channel Effects for the Bottomonium with Realistic Wave Functions

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Abstract

Using the Gaussian expansion method (GEM), we employ realistic wave functions to calculate coupled-channel effects for bottomonium within the framework of the 3P0 model. The simplicity and accuracy of GEM are explained. We calculate mass shifts, probabilities of the B meson continuum, S-D mixing angles, and strong and dielectric decay widths. Our calculation shows that both S-D mixing and the B meson continuum can contribute to the suppression of the vector meson' s dielectric decay width. We suggest more precise measurements of the radiative decays of $\Upsilon(10580)$ and $\Upsilon(11020)$ to distinguish these two effects. The above quantities are also calculated using the simple harmonic oscillator (SHO) wave function approximation for comparison. The deviation between GEM and SHO indicates that it is essential to treat the wave functions accurately for near-threshold states.

Introduction

Heavy quarkonium is a multi-scale system covering all regimes of quantum chromodynamics (QCD), making it an ideal place to study strong interactions [1]. Despite the success of QCD in the high-energy region, due to asymptotic freedom, non-perturbative effects dominate at low energies and cause problems for perturbative calculations. One tool to study these non-perturbative effects is lattice QCD. However, due to its enormous computational requirements, it is still unable to calculate all physical quantities with current computing power. Another important approach is to develop various phenomenological models. Among these, the quark model is a prominent one. Under the quark model framework, various types of interactions have been suggested by different groups, achieving many impressive successes (see e.g. Refs. [2-6]). However, these potential models cannot be the whole story. One important missing ingredient is the mechanism to generate quark-antiquark pairs, which enlarges the Fock space of the initial state, i.e., the initial state contains multi-quark components.

These multi-quark components will change the Hamiltonian of the potential model, causing mass shifts and mixing between states with the same quantum numbers, or directly contributing to open-channel strong decay if the initial state is above threshold. These consequences can be summarized as unquenched effects or coupled-channel effects. Coupled-channel effects were considered at least 30 years ago by Törnqvist et al. in Refs. [7-11], who extended the quark model to an unquenched quark model.

Despite the fact that the underlying quark pair creation mechanism is not fully understood even now, there are still different phenomenological models to decode the mystery, such as the 3P0 model [12-14], the flux-tube breaking model [15,16], and microscopic decay models [4,17,18]. Among these, the simplest and most successful one is the 3P0 model, where the generated light quark pair shares the same quantum numbers as the vacuum.

Even though the 3P0 model has been extensively studied by many researchers, almost all calculations use the SHO wave function approximation to simplify the calculation (see e.g. [18–26]). A simple yet powerful method to handle the wave function precisely is still not widely known. We propose using the Gaussian expansion method (GEM) [27] to accurately evaluate the wave function convolution.

There have already been some works related to GEM. In Refs. [28–41], GEM is adopted to calculate wave functions using the variational method approach. Coupled-channel effects with GEM have only been studied for some specific cases, such as X(3872) and P-wave Ds mesons [42, 43]. In Refs. [44–46], the authors also use GEM to calculate the spectrum and open-channel strong decays of light mesons and some specific charmonia, where the coupled-channel-induced mass shift is not considered. Even though mass shifts can be partly absorbed by redefining the potential, the potential model cannot describe near-threshold effects [47]. We want to emphasize that mass shifts and open-channel strong decays are directly correlated by coupled-channel effects, so it is essential to evaluate them under the same framework and calculate them precisely.

So far, a precise evaluation and thorough discussion of the coupled-channel effects are still missing, and the validity of the SHO approximation is yet to be clarified. In this paper, we fill these gaps by providing a thorough discussion of coupled-channel effects for bottomonium and also predict some important results on the dielectric and radiative decays of vector mesons that will be tested by experiment.

The paper is organized as follows. In Sec. 2, we explain the details of the Cornell potential and 3P0 model, where we deduce the formulas for mass shift, open-channel strong decay width, and S-D mixing. Sec. 3 focuses on the calculation details and GEM, where the advantages of GEM are elucidated and the procedure to fit the wave function is explained. Sec. 4 is devoted to discussing the possible impacts of coupled-channel effects for bottomonium on the spectrum, open-channel strong decays, probabilities of the B meson continuum, S-D mixing, and the vector meson' s dielectric and radiative decays. We also explicitly show the deviation between GEM and SHO approximation. Finally, we give a short summary of this work in Sec. 5.

2.1 Cornell Potential Model

As the quenched limit, the wave functions for heavy quarkonium are obtained by solving the Schrödinger equation with the well-known Cornell potential [4, 17]

$$V(r) = -\frac{\alpha}{r} + \lambda r + c,$$

where α, λ, and c stand for the strength of the color Coulomb potential, the strength of linear confinement, and mass renormalization, respectively. To restore the hyperfine or fine structures of bottomonium, we use the following form of the spin-dependent interactions

$$V_s(r) = \left(\frac{2\alpha}{m_b^2 r^3} - \frac{\lambda}{m_b^2 r}\right)\mathbf{L}\cdot\mathbf{S} + \frac{2\alpha}{3m_b^2}\tilde{\delta}(r)\mathbf{S}_b\cdot\mathbf{S}_{\bar{b}} + \frac{\lambda}{m_b^2 r}\left[\mathbf{S}_b\cdot\mathbf{S}_{\bar{b}} - 3(\mathbf{S}_b\cdot\hat{\mathbf{r}})(\mathbf{S}_{\bar{b}}\cdot\hat{\mathbf{r}})\right],$$

where $\mathbf{L}$ denotes relative orbital angular momentum, $\mathbf{S} = \mathbf{S}_b + \mathbf{S}_{\bar{b}}$ is the total spin of the b quark pairs, and m_b is the b quark mass. Since the non-relativistic expansion fails if two composite quarks are very close to each other, instead of the Dirac delta function in the second term, we use the smeared delta function, which can be written as $\tilde{\delta}(r) = (\sigma/\sqrt{\pi})^3 e^{-\sigma^2 r^2}$ [47, 48]. The Hamiltonian of the Schrödinger equation in the quenched limit is represented as

$$H_0 = 2m_b + \frac{\mathbf{p}^2}{2m_b} + V(r) + V_s(r).$$

We treat the spin-dependent term as a perturbation, and the spatial wave functions are obtained by solving the Schrödinger equation numerically using Numerov’ s method [49].

2.2 3P0 Model and Coupled-Channel Effects

For the coupled-channel calculation, we adopt the widely used 3P0 model or quark pair creation model, which was first proposed by L. Micu [12] in 1969 and then extended by A. Le Yaouanc et al. in the 1970s [13,14].

In this model, the generated quark pairs have vacuum quantum numbers $J^{PC} = 0^{++}$. After simple arithmetic, one can conclude that the relative orbital angular momentum and total spin are both equal to 1, and the total angular momentum is 0. In the notation of ${}^{2S+1}L_J$, one should write it as ${}^3P_0$, which explains the model’ s name.

The interaction Hamiltonian can be expressed as

$$H_I = 2m_q\gamma \int d^3x\, \bar{\psi}_q\psi_q,$$

where m_q is the produced quark mass and γ is the dimensionless coupling constant. Since the probability to generate heavier quarks is suppressed, we use the effective strength $\gamma_s = m_q\gamma$ in the following calculation, where $m_q = m_u = m_d$ is the constituent quark mass of the up (or down) quark and m_s is the strange quark mass.

Figure 1 [Figure 1: see original paper]: Sketch of coupled-channel effects in the 3P0 model. i and f respectively denote the initial and final states with the same J^{PC}, and $B\bar{B}$ stands for all possible B meson pairs.

The 3P0 Hamiltonian induces not only open-flavor strong decays of heavy quarkonium above threshold but also coupled-channel effects. As sketched in Fig. 1, the experimentally observed state should be a mixture of a pure quarkonium state (bare state) and B meson continuum. Put into formula, the physical or experimentally observed state $|A\rangle$ should be expressed as

$$|A\rangle = c_0|\psi_0\rangle + \int d^3p \sum_{BC} c_{BC}(p)|BC;p\rangle,$$

where c_0 and c_{BC} stand for the normalization constants of the bare state and B meson continuum, respectively. $|\psi_0\rangle$ is normalized to 1, and $|A\rangle$ is also normalized to 1 if it lies below the $B\bar{B}$ threshold. $|BC;p\rangle$ is normalized as $\langle BC;p_1|B'C';p_2\rangle = \delta^3(p_1-p_2)\delta_{BB'}\delta_{CC'}$, where p is the momentum of the B meson in $|A\rangle$' s rest frame.

Combining the Cornell potential and the dynamics of quark pair generation, we get the full Hamiltonian

$$H = H_0 + H_{BC} + H_I,$$

with the following relations:

$$H_0|\psi_0\rangle = M_0|\psi_0\rangle,$$

$$H_0|BC;p\rangle = 0,$$

$$H_{BC}|\psi_0\rangle = 0,$$

$$H_{BC}|BC;p\rangle = E_{BC}|BC;p\rangle,$$

$$H|A\rangle = M|A\rangle,$$

where M_0 is the bare mass of the charmonium state and can be solved directly from the Schrödinger equation.

The interaction between B mesons is neglected, so the energy of the meson continuum can be expressed as

$$E_{BC} = \sqrt{m_B^2 + p^2} + \sqrt{m_C^2 + p^2}.$$

When Eq. (11) is projected onto each component, we immediately get

$$\langle\psi_0|H|\psi\rangle = c_0M = c_0M_0 + \int d^3p\, c_{BC}(p)\langle\psi_0|H_I|BC;p\rangle,$$

$$\langle BC;p|H|\psi\rangle = c_{BC}(p)M = c_{BC}(p)E_{BC} + c_0\langle BC;p|H_I|\psi_0\rangle.$$

Solving c_{BC} from Eq. (13) and substituting back into Eq. (12), then eliminating c_0 on both sides, we get an integral equation where $M = M_0 + \Delta M$,

$$\Delta M = \int d^3p \sum_{BC} \frac{|\langle BC; p|H_I|\psi_0\rangle|^2}{M - E_{BC} - i\epsilon}.$$

The sum over BC is restricted to the ground-state $B(s)$ mesons, i.e., $B\bar{B}$, $B\bar{B}^* +$ h.c., $B^*\bar{B}^*$, $B_s\bar{B}_s$, $B_s\bar{B}_s^* +$ h.c., $B_s^*\bar{B}_s^*$. Note that the $i\epsilon$ term is added to handle the situation when $m_A > m_B + m_C$. In this case,

$$\mathrm{Im}(\Delta M) = \pi \sum_{BC} |\langle BC; p_B|H_I|\psi_0\rangle|^2,$$

which equals one half of the decay width. p_B and E_B respectively denote the momentum and energy of the B meson.

The wave function overlap integration lies in the term

$$\langle BC; p_B|H_I|\psi_0\rangle = \sum_{\text{polarization}} \int d^3k\, \phi_0(\mathbf{k}+\mathbf{p}_B)\phi_B^*(\mathbf{k}+x_B\mathbf{p}_B)\phi_C^*(\mathbf{k}+x_C\mathbf{p}_B)|\mathbf{k}|Y_1^m(\theta_{\mathbf{k}}, \phi_{\mathbf{k}}),$$

where $x_B = m_4/(m_1+m_4)$, $x_C = m_3/(m_2+m_3)$, and $m_1 = m_2 = m_Q$, $m_3 = m_4$ respectively denote the b quark and the light quark mass.

Once M is solved, the coefficients of different components can be worked out as well. For states below threshold, the normalization condition $|A\rangle$ can be rewritten as

$$|c_0|^2 + \int d^3p|c_{BC}|^2 = 1.$$

After substitution of c_{BC}, we get the probability of the $b\bar{b}$ component

$$P_{b\bar{b}} := |c_0|^2 = 1\Big/\left(1 + \int_0^\infty dp\, p^2 \sum_{BC} \frac{|M_{LS}|^2}{(M - E_{BC})^2}\right),$$

where $|M_{LS}|^2$ is represented as

$$|M_{LS}|^2 = \int d\Omega_B|\langle BC; p_B|H_I|\psi_0\rangle|^2.$$

2.3 Coupled Channel Induced S-D Mixing

From the quark model' s perspective, the spatial wave functions of the $J^{PC} = 1^{--}$ family can be both S- and D-wave. It is natural to expect that the experimentally observed vector states are mixtures of S- and D-waves. As in the case of conventional meson coupling with $B\bar{B}$ continuum, we rewrite it into a matrix

$$\begin{pmatrix} M_0 & \int d^3p\langle\psi_0|H_I|BC\rangle \\ \int d^3p\langle BC|H_I|\psi_0\rangle & \int d^3p\langle BC|H_{BC}|BC\rangle \end{pmatrix} \begin{pmatrix} c_0 \\ c_{BC} \end{pmatrix} = M \begin{pmatrix} c_0 \\ c_{BC} \end{pmatrix},$$

where the integration part should be understood as formal notation, and one needs to insert all the p-dependent parts into the integral. For example, in the above case, one may naively get the following form after diagonalization

$$(M - M_0)(M - E_{BC}) = \int d^3p|\langle\psi_0|H_I|BC\rangle|^2.$$

However, the correct form should be understood as Eq. (15), where the $(M - E_{BC})$ term has moved inside the integration.

The advantage of the matrix form is that one can easily see its structure and generalize it to the S-D mixing case. Under the assumption that $(n+1)S$ mixes only with nD, we have

$$\begin{pmatrix} M_S^0 + \Delta M_S & \Delta M_{SD} \\ \Delta M_{SD} & M_D^0 + \Delta M_D \end{pmatrix} \begin{pmatrix} c_S \\ c_D \end{pmatrix} = M \begin{pmatrix} c_S \\ c_D \end{pmatrix},$$

where

$$\Delta M_f = \int d^3p \sum_{BC} \frac{|\langle\psi_f|H_I|BC\rangle|^2}{M - E_{BC} - i\epsilon} \quad (f = S, D),$$

$$\Delta M_{SD} = \int d^3p \sum_{BC} \frac{\langle\psi_S|H_I|BC\rangle\langle BC|H_I|\psi_D\rangle}{M - E_{BC} - i\epsilon}.$$

The S-D mixing induced by the tensor part of the potential is so small, typically around 0.8° in our calculation (see also Appendix A in Ref. [50]), so it is quite reasonable to set $H_T = 0$. After this approximation, one can reexpress c_{BC} in terms of c_S, c_D and easily get

$$\begin{pmatrix} M_S^0 + \Delta M_S & \Delta M_{SD} \\ \Delta M_{SD} & M_D^0 + \Delta M_D \end{pmatrix} \begin{pmatrix} c_S \\ c_D \end{pmatrix} = M \begin{pmatrix} c_S \\ c_D \end{pmatrix}.$$

From the above equation, both the mass and the relative ratio c_S/c_D can be worked out. For states below threshold, the probability can be solved once the mass is known, which is a generalization of Eq. (19):

$$|c_S|^2+|c_D|^2+\int d^3p\sum_{BC}\frac{1}{(M-E_{BC})^2}\left(|c_S|^2H^2_{S,BC}+|c_D|^2H^2_{D,BC}+2\mathrm{Re}[c_Sc_D^*H_{S,BC}H_{BC,D}]\right)=1,$$

where $H_{f,i}$ stands for $\langle f|H_I|i\rangle$.

One will get a complex solution $M=M_{BW}+i\Gamma/2$ if $M_{BW}>m_B+m_{\bar{B}}$, where M_{BW} represents the Breit-Wigner mass of the resonance, and Γ is the decay width after considering S-D mixing. As a cross-check, one can also calculate the decay width directly with the following formula:

$$\Gamma_{SD}=2\left(|c_S|^2\mathrm{Im}(\Delta M_S)+|c_D|^2\mathrm{Im}(\Delta M_D)+2\mathrm{Re}(c_S^*c_D\mathrm{Im}(\Delta M_{SD}))\right).$$

Eq. (24) is much more difficult to solve than Eq. (15). The method we use to solve this equation will be discussed in the next section.

3.1 Parameter Selection

As a first step, we tune the wave functions to be consistent with the dielectric decay widths of $\Upsilon(nS)$ for $n\leq 3$. Theoretically, dielectric decay widths can be expressed as [51-54]

$$\Gamma_{ee}=\beta\frac{4\alpha^2e_b^2}{M^2}|c_SR_{nS}(0)+c_DR''_{nD}(0)|^2,$$

where $\beta=(1-16\alpha_s/3\pi)$ is the QCD radiative correction, and $e_b=-1/3$ is the b quark charge in units of electron charge. $R_{nS}(0)$ denotes the radial S-wave function at the origin, and $R''_{nD}(0)$ is the second derivative of the radial D-wave function at the origin. c_S and c_D respectively denote the normalization coefficients before the S- and D-waves. Note that from the perspective of coupled channels, they are not restricted to be real-valued and $|c_S|^2+|c_D|^2\neq 1$. For below-threshold states, the correct normalization is given by Eq. (27). Nevertheless, if the imaginary part of ΔM is neglected in Eq. (24), the corresponding solutions will be real, and one can easily get a feel for how big the mixing is by defining $\tan\theta:=|c_S/c_D|$ for S-wave-dominated states and $\tan\theta:=|c_D/c_S|$ for D-wave-dominated states.

There is also an argument that higher-order QCD corrections may be important [50]; thus β has to be treated as an effective constant. So in order to reduce parameter uncertainty, we tune the wave functions to reproduce $\Gamma_{ee}(nS)/\Gamma_{ee}(1S)$ for $n=2,3$ (see Fig. 6 [Figure 6: see original paper]).

Table 1 : The parameters used in our calculation. Due to the implicit treatment of color and flavor degrees of freedom, these factors do not appear in our calculations.

Parameter	Value
α	0.34
λ	0.22 GeV²
c	0.435 GeV
m_b	4.5 GeV
$m_u = m_d$	0.33 GeV
m_s	0.5 GeV
γ	0.205
σ	3.838 GeV

3.2 Gaussian Expansion Method

There are at least two ways to solve Eq. (15). The first is the recursion method, which is based on the observation that the mass shift is expected to be small compared with the bare mass, i.e., set $M = m_0$ as the first step, perform the integration in Eq. (15) to get the mass shift, then set $M = m_0 + \Delta m$ again, and so on until the result converges. One can even make a further approximation and only perform the first recursion step. However, this method only applies to the single-channel mass shift formula (15). In S-D mixing cases, such as Eq. (24), the mass difference between $M_S^0 + \Delta M_S$ and $M_D^0 + \Delta M_D$ is small, so even a small error in the off-diagonal term in Eq. (24) will ruin the prediction of the S-D mixing angle.

The second way is to solve the equation by brute force, i.e., for the energy ranges we are interested in, work out all the integrations in Eq. (15) or Eq. (24) at specific energy points. We use this method despite its huge computational cost. The benefit is that we can extract a lot of information about the wave function' s impact on the mass shift. One can also change the 3P0 coupling constant γ or the mass renormalization constant c to see the possible consequences.

High-precision work will not be convincing if there is no way to precisely evaluate the integration, which has a key ingredient—the wave function. One can indeed evaluate the amplitude purely numerically, as the authors do in Refs. [47, 55]; however, we still want analytic expressions, which are more convenient if we want to change the parameters and repeat the calculations.

To achieve that, various groups approximate the wave functions by simple harmonic oscillators (SHOs) (see e.g. [18–20, 22, 23]). The oscillator parameters β are usually determined by requiring that the root-mean-square radii be equal to those of the initial states [56–58] or by maximizing their overlap with the numerical wave function [18].

Figure 2 [Figure 2: see original paper]: Comparison of $\Upsilon(4S)$' s spatial wave function. Numerical values and GEM fit are denoted by black dots and red solid curve, respectively. Black dashed and solid curves represent single SHO approximation by matching $\langle r \rangle$ and maximizing wave function overlap, respectively.

To improve accuracy, people also expand the true wave function in terms of SHOs (see e.g. Refs. [59, 60]). As a consequence, one gets fairly complicated analytic expressions for highly excited states, such as expression (2.12) in Ref. [59]. Due to the highly oscillatory behavior of the excited SHOs, one would need a large number of SHOs to achieve ideal precision.

We think it is necessary to fully respect the wave function and make a precise calculation of the transition amplitude, which is shown to be essential for states near threshold. In this work, we obtain both analytic expressions and high precision by using the Gaussian expansion method (GEM) proposed by Hiyama et al. [27]. This method is based on the observation that a bound state' s wave function can be expanded in Gaussian bases as follows:

$$\psi_{NLM}(\mathbf{r}) = \sum_{n=1}^{n_{\max}} c_n e^{-(r/\beta_n)^2} r^L Y_{LM}(\theta, \phi),$$

where β_n and c_n denote oscillator parameters and corresponding coefficients, respectively, and $n_{\max}$ is the number of Gaussian basis functions. In this work, $n_{\max} = 5 \sim 20$ for initial states and 5 for B mesons, with β_n lying in the range $0.1 \sim 5$ GeV.

Compared with the SHO basis, the Gaussian basis is no longer orthogonal. So a little trick is used to speed up the fitting procedure. As explained in Ref. [27], the β_n are set to be a geometric series. Instead of increasing the number of SHO basis functions for a fixed β, GEM can both increase the basis number and change β to improve the fit. The quality of the fitting by GEM is quite impressive, as shown in Fig. 2.

Because the 3P0 model calculation is easily done in momentum space, we need to make a Fourier transformation of the position-space wave function. One benefit of the SHO wave function is that it is invariant after Fourier transform apart from the substitution $\beta \to 1/\beta$. Since the Gaussian basis is the ground-state SHO wave function, it naturally keeps this property. This means that after fitting position-space wave functions, we can rebuild the momentum-space wave functions by

$$\psi_{NLM}(\mathbf{p}) = \sum_{n=1}^{n_{\max}} c_n e^{-(p\beta_n)^2} p^L Y_{LM}(\theta, \phi).$$

What makes GEM simple is that there are a minimum number of polynomials in the integration, which simplifies the expression from the beginning. GEM is quite universal and is not limited to wave functions obtained by solving the non-relativistic Schrödinger equation.

To work out the analytic expression, one has to deal with the associated Laguerre polynomials if SHO wave functions are involved, as well as sophisticated angular integration. Although Ref. [61] shows the general method to perform the integration, and there are analytical expressions [59], these expressions are quite lengthy and they only apply to the 3P0 model.

The complexity can be bypassed if we transfer the spherical harmonics into Cartesian form [62] from the beginning. After this transformation, the general form of the integration can be compactly expressed as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp_x dp_y dp_z \ \exp(-\mathbf{p}^T\cdot\mathbf{A}\cdot\mathbf{p}-\mathbf{B}\cdot\mathbf{p}-C)f(p_x,p_y,p_z),$$

where $\mathbf{p}$, $\mathbf{A}$, $\mathbf{B}$, and C in the exponent denote $(p_x,p_y,p_z)^T$, a 3×3 real symmetric matrix, a three-vector, and a constant, respectively. $f(p_x,p_y,p_z)$ is nothing but a polynomial, so Eq. (32) is a standard form of Gaussian integration that can be easily done even manually.

After the integration is performed, we can easily transform it back to the spherical basis by the substitution $p_x = P\sin\theta\cos\phi$, $p_y = P\sin\theta\sin\phi$, $p_z = P\cos\theta$. Another benefit of this transformation is that it can easily handle much more complicated polynomials $f(p_x,p_y,p_z)$ that may appear in other quark pair creation models.

4 Result and Discussion

4.1 Mass Shift and Open Channel Strong Decay

From Tables 2 and 3, one can find that the mass shifts are generally the same between GEM and SHOs, with a few exceptions for near-threshold states. The mass shifts in the same multiplet are also almost equal simply because their wave functions are identical and their bare masses are approximately equal. This conclusion is consistent with the loop theorem in Ref. [63].

For states below threshold, ΔM values are all negative, and the closer to the threshold, the greater the mass reduction. With GEM, this conclusion holds even for states slightly above threshold. This differs from Refs. [22, 23], where SHOs are used to calculate the mass shift. Taking the h_b family as an example, our mass shift grows as the mass increases, regardless of whether we use GEM or SHO; however, in Ref. [22] and Ref. [23], the largest mass shift occurs for $h_b(1P)$ and $h_b(2P)$, respectively.

For states above threshold, the mass shift behavior becomes complicated (see Fig. 4), and it is not appropriate to draw the conclusion that the mass shift of a state above the $B\bar{B}$ threshold is positive. This conclusion is only true for asymptotically large mass, and in that case, $\Delta M \propto 1/M$. We should also point out that this mathematical fact does not mean it will definitely happen. The reason is that when the mass becomes larger, more B meson channels contribute, and one cannot determine the sign of ΔM before summing all possible channels' contributions in Eq. (15).

To study this sensitivity, we also plot the dependence of ΔM and decay width on the initial state mass for vector mesons above threshold in Figs. 4 and 5 [Figure 5: see original paper], respectively. As a concrete example, one can see this sensitivity by comparing $\Upsilon(4S)$ with $\Upsilon(6S)$. Compared with $\Upsilon(4S)$, the wave function of $\Upsilon(6S)$ has more nodes; however, based on this fact alone one cannot conclude that ΔM' s behavior is more complicated. The important reason is that the bare mass of $\Upsilon(6S)$ is also farther from threshold, causing averaging of the wave function overlap integration in Eq. (17). Note also that the absolute value of the mass shift of $\Upsilon(6S)$ calculated with SHO is larger than with GEM; however, in the $\Upsilon(4S)$ case, we have the opposite conclusion if we choose the lowest intersection point of ΔM and $M - M_0$.

From Table 2, one can find that the masses predicted in Ref. [22] are generally closer to the experimental data. However, we want to stress that the spectrum is an important but not the only criterion to judge whose parameters are better. As shown in Fig. 6, the dielectric decay ratios $\Gamma_{ee}/\Gamma_{ee}(1S)$ calculated with the parameters given in Ref. [22] are generally smaller than experimental measurements before coupled-channel effects are taken into account. As will be discussed in Sec. 4.2, coupled-channel effects will suppress rather than enhance these ratios, so their parameters are difficult to explain the dielectric decays of vector mesons despite their success in the spectrum.

Table 2 : Total mass shift (in MeV) induced by coupled-channel effects. M_0 denotes the bare mass of the Cornell potential whose parameters are shown in Table 1. M_{theory} is the mass after considering coupled-channel effects. The last two columns of $-\Delta M$ and M_{theory} are taken from Ref. [22] and Ref. [23], whose parameters are different from ours. M_{exp} denotes the experimentally measured value. For simplicity, the experimentally measured $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ are assumed to be $\Upsilon(4S)$, $\Upsilon(5S)$, and $\Upsilon(6S)$, respectively. "–" represents that the corresponding value is not available.

Table 3 : The mass shift (in MeV) of every coupled channel. Coupled-channel-induced S–D mixing is not considered in this table. "0" represents that the contributions of some channels are forbidden. For simplicity, an overall negative sign has been omitted for all channels. Note that for a few channels, mass shifts are positive.

Figure 4: The dependence of ΔM of $\Upsilon(nS)$ and $\Upsilon_1(nD)$ families on the mass of the initial states. GEM and SHO results are denoted by red solid curve and black dashed curve, respectively. $M - M_0$ is shown by the solid black line. One can read the M values (which are shown in Table 2) and corresponding ΔM from the intersection points of $M - M_0$ and ΔM.

The complicated structure of the mass shift of $\Upsilon(4S)$ needs further discussion. Even though the curves of GEM and SHO share some common features, the small difference is sufficient to generate a large discrepancy in the mass shift. Another interesting feature of this plot is that GEM has three solutions, implying that more resonances may appear compared with potential model predictions.

This sensitivity can also be seen in the decay width behavior in Fig. 5. Over a large energy range 10.58 $\sim$ 10.73 GeV, the decay width of $\Upsilon(4S)$ calculated by GEM can be around two times as large as that from SHO. Our decay width plot of $\Upsilon(4S)$ also shares some resemblance with Fig. 2 in Ref. [64], where the prediction of SHO is not calculated.

The deviations in mass shift and decay width tell us that it is necessary to adopt realistic wave functions rather than SHO approximation in coupled-channel calculations.

Of course, one may argue that since the bare mass is 140 MeV heavier than the experimental measurement, if the bare mass is tuned closer to the $B\bar{B}$ threshold, the difference between GEM and SHO would be small and we would get one solution. However, we want to stress that the bare mass is directly related to the wave function; in the case where we have a smaller bare mass, the wave function will also be different, thus causing different mass shift behavior. This sensitivity also reminds us that taking only a one-step approximation in the recursive method to solve Eq. (15) may cause a large error, so an accurate treatment of the wave function and a precise method to solve Eq. (15) are essential for near-threshold states.

Figure 5 [**Figure 5: see original paper**]: The dependence of the open-channel strong decay widths of $\Upsilon(S)$ and $\Upsilon_1(D)$ families on the mass of the initial states. GEM and SHO results are denoted by red solid curve and black dashed curve, respectively. One can directly read the total decay width from this plot.

Table 4 : Probabilities of every coupled channel and $b\bar{b}$ component for states below threshold. The effect of S-D mixing is not considered in this table. "0" represents that the contributions of some channels are forbidden. The overall % has been omitted for simplicity. Note that despite $m(h_b(3P))$ calculated with GEM and SHO and $m(\chi_{b1}(3P))$ calculated with SHO being above the $B\bar{B}$ threshold, they do not couple to $B\bar{B}$ and their masses are still smaller than $B\bar{B}^*$, so the probabilities of B meson continuum are well-defined.

If $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ are treated as pure S- or D-wave states, we obtain the open-channel decay widths shown in Table 5. It is worth noting that this assumption is oversimplified, so the absolute values should not be taken too seriously.

Table 5 : Open-channel strong decay widths (in MeV) of pure S- and D-wave vector bottomonia. $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ are considered to be close to $4S \sim 3D$, $5S \sim 4D$, and $6S \sim 5D$, respectively.

4.2 S-D Mixing and Dielectric Decay

As explained in Sec. 2 and sketched in Fig. 1, coupled-channel effects will also induce mixing among states with the same J^{PC}. In this paper, we focus on the mixing between the $\Upsilon(S)$ and $\Upsilon_1(D)$ families. The Cornell potential model tells us that the mass splitting between $\Upsilon((n+1)S)$ and $\Upsilon_1(nD)$ is smaller than other configurations, such as $\Upsilon(nS)$ and $\Upsilon((n+1)S)$ or $\Upsilon_1(nD)$ and $\Upsilon_1((n+1)D)$ states, so it is quite reasonable to assume that mixing only occurs between $\Upsilon((n+1)S)$ and $\Upsilon_1(nD)$. The masses and corresponding mixing angles after considering S-D mixing are listed in Table 6.

In Eq. (24), the overall phase before c_S and c_D is nonphysical, so we are free to set the phase of c_D to be 0, i.e., $c_D \geq 0$. Under this convention and the normalization condition Eq. (27), the ratio c_S/c_D is adequate to fix the values of c_S and c_D. If the imaginary part of ΔM_f and ΔM_{SD} in Eq. (24) is neglected, one would get real solutions both for M and c_S/c_D. After this approximation, one can deduce the mixing angles. However, the definition of c_S and c_D doesn' t exist for states above threshold [56, 65]. Despite this difficulty, we follow Ref. [7], assuming that these open channels' contributions are neglected. Under this assumption, quantities related to S-D mixing are shown in Table 6.

Table 6 : Mixing between $\Upsilon((n+1)S)$ and $\Upsilon_1(nD)$ calculated with GEM and SHO. The unit of mass is GeV. M_0 is the bare mass calculated by the potential model. M_{pure} is taken from column 7 in Table 2, where S-D mixing is not considered. M_{comp} and M_{real} both denote the masses after S-D mixing; the difference is that the latter is the solution when one neglects the imaginary part of Eq. (24), while the former is the precise solution of Eq. (24), and its imaginary part equals one half of the decay width. c_S/c_D(comp) and c_S/c_D(real) denote the ratio c_S/c_D corresponding to M_{comp} and M_{real}, respectively.

From Table 6, we find that the masses barely change after considering S-D mixing for states below threshold, indicating that the mixing angles are approximately zero. So it is reasonable to treat $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ as pure S-wave states. This conclusion also agrees with the loop theorem in Ref. [63]. From Eq. (29), we also learn that the dielectric decay of $\Upsilon_1(D)$ is suppressed by the b quark mass m_b, so the small mixing also provides a natural explanation for why these D-wave vector mesons are difficult to find at e^+e^- colliders. From Table 6, one can also read off the open-channel strong decay widths after considering S-D mixing for states above threshold. However, one cannot compare the imaginary part of M_{comp} directly with experimental data because its real part (which is the Breit-Wigner mass) does not equal the experimental mass, thus their phase space for $B\bar{B}$ is different from experiment.

A natural and direct consequence of non-negligible S-D mixing is the suppression of $\Gamma_{ee}(S)$ or enhancement of $\Gamma_{ee}(D)$. As can be seen from Fig. 6, the dielectric decay width of $\Upsilon(10580)$ and $\Upsilon(11020)$ is highly suppressed experimentally. Under the assumption that $\Upsilon(10580)$ and $\Upsilon(11020)$ are S-wave-dominated states, one may be tempted to introduce a large S-D mixing angle for these highly excited states (see e.g. Ref. [50]).

The unexpected large central value of $\Gamma_{ee}(\Upsilon(10860))$ seems to favor a small S-D mixing angle; however, due to its large errors, a mixing angle as large as 27° can also reproduce the data, which gives a decay width at the lower bound [50].

Of course, more precise measurement of $\Upsilon(10860)$' s dielectric decay will tell us whether the claim of large S-D mixing is correct or not, if S-D mixing is fully responsible for this suppression.

As can be seen from Table 6, except for the $4S-3D$ case, which will be discussed shortly, we get rather small mixing angles not only for below-threshold states but also for highly excited states. It seems that the coupled-channel formalism cannot explain the suppression of Γ_{ee}. However, we want to point out that S-D mixing is not the only way to suppress $\Gamma_{ee}(S)$. Even though the S-D mixing angles are small, Γ_{ee} can still be suppressed by the B meson continuum.

Figure 6 [Figure 6: see original paper]: Comparison of $\Gamma_{ee}/\Gamma_{ee}(1S)$ between different models. Results of the Cornell potential with our parameters and parameters of Ref. [22] are respectively represented by blue regular and gray inverted triangles. Red dots and black rectangles respectively denote the predictions of Eq. (24) with and without neglecting the imaginary part. Black dots with error bars are values taken from PDG [66].

The suppression due to the B meson continuum is not difficult to understand. If conventional mesons have non-negligible components of B meson pairs, these meson pairs must undergo one more 3P0 vertex before annihilating into e^+e^- pairs. Since the Hamiltonian is small compared with the Cornell potential, it is reasonable to discard the contribution of these meson pairs, then Γ_{ee} is suppressed. This suppression is universal for both S- and D-wave vector mesons, in contrast to S-D mixing, which enhances $\Gamma_{ee}(D)$.

We take into account both S-D mixing (see Table 6) and the B meson pair suppression mechanism (see Table 4) in this work, and the results of $\Gamma_{ee}/\Gamma_{ee}(1S)$ are shown in Fig. 6.

From Fig. 6, one can see that the coupled-channel results agree well with experiment except for $\Upsilon(11020)$. The large suppression of $\Gamma_{ee}(\Upsilon(10580))$ deserves more explanation. From the quark model' s perspective, $\Upsilon(10580)$ is suggested to be a $4S$ or $3D$ state. If it is S-wave-dominated, the mixing angle is about 9°, which is still not large enough to reproduce Γ_{ee}. In fact, according to our calculation, the major suppression comes from the B meson pairs.

Even though $\Upsilon(10580)$ is above the $B\bar{B}$ threshold, it is still below and quite close to the $B\bar{B}^*$ threshold. As $h_b(3P)$, $\chi_{b0}(3P)$, and $\chi_{b0}(3P)$ show us in Table 4, being closer to the threshold means larger probabilities of B meson continuum. Like the authors did in Ref. [7], we neglect the $B\bar{B}$ probability and work out the probabilities of other channels. The probabilities of $B\bar{B}^* + \text{h.c.}$, $B^*\bar{B}^*$, $B_s\bar{B}_s$, $B_s\bar{B}_s^* + \text{h.c.}$ are 20.06%, 11.7%, 0.125%, 0.37%, and 0.51%, respectively, which means $P_{b\bar{b}} = 67.2\%$. So as an estimation, one will get only two-thirds of the decay width predicted by the potential model.

For 3D-dominated states with Breit-Wigner mass 10.731 GeV in Table 6, its large mixing angle may attract one' s attention. Because it is more difficult to generate at e^+e^- colliders compared with 4S-dominated states, and its mass is

80 MeV heavier than the 4S-dominated states, we do not consider it as $\Upsilon(10580)$. In the $4S-3D$ mixing case, because of the oscillatory behavior of ΔM_f and $\langle\psi_f|H_I|\psi_i\rangle$, there is one more pair of solutions of M in Eq. (24) with GEM. For the 4S-dominated state, $M_{\text{comp}} = 10.673 + 0.0989i$, $M_{\text{real}} = 10.675$, $\theta = 18.18^\circ$, and for the 3D-dominated state, $M_{\text{comp}} = 10.718 + 0.0441i$, $M_{\text{real}} = 10.7233$, $\theta = 18.4^\circ$. For the same reasons, we also do not consider it as $\Upsilon(10580)$.

Another interesting detail of the B meson continuum is the slightly increased ratio of $\Gamma_{ee}/\Gamma_{ee}(1S)$. The mixing angles of $5S-4D$ and $6S-5D$ are so small that Γ_{ee} barely changes; nevertheless, due to the small B meson continuum component of $\Upsilon(1S)$, $\Gamma_{ee}(1S)$ will be suppressed by about 0.013, and as a consequence, the ratio $\Gamma_{ee}/\Gamma_{ee}(1S)$ becomes slightly larger after considering coupled-channel effects.

For $\Upsilon(10876)$ and $\Upsilon(11020)$, all ground-state B meson channels are open. We can no longer deduce the probabilities of these B meson pairs with Eq. (24). There is no B meson continuum suppression in this work. This can reproduce the dielectric decay width of $\Upsilon(10876)$ but not $\Upsilon(11020)$.

As shown in Fig. 6, there is a notable discrepancy between our calculation and experiment for $\Upsilon(11020)$' s dielectric decay width. This issue may come from two assumptions we use to simplify the calculation. First, we only consider mixing between $6S$ and $5D$. In fact, with increasing radial quantum number, the energy levels of S- or D-wave states become denser, so mixing may exist between several S- and D-wave states. Second, the probabilities of excited B meson pairs are neglected. This may also cause problems. For example, $\Upsilon(11020)$ is only 26 MeV lighter than $m_{B^*}+m_{B_1}$, so a large suppression of $P_{b\bar{b}}$ is naturally expected, causing a large suppression of the dielectric decay width.

It is possible to distinguish S-D mixing and B meson pair suppression mechanisms by measuring radiative decays. Theoretically, the E1 transition can be represented by [54, 67, 68]

$$\Gamma(n^{2S+1}L_J \to n'^{2S'+1}L'_{J'} + \gamma) = C_{fi}\delta_{SS'}e_b^2\alpha|\langle f|r|i\rangle|^2E_\gamma^3,$$

where $e_b = -1/3$, α and E_γ respectively denote the fine structure constant and the energy of the emitted photon. $\langle f|r|i\rangle$ and C_{fi} are represented by

$$\langle f|r|i\rangle = \int_0^\infty R_f(r)R_i(r)r^3dr,$$

$$C_{fi} = \frac{\max(L,L')(2J'+1)}{2J+1}\begin{Bmatrix} L' & J' & S \\ J & L & 1 \end{Bmatrix}^2.$$

From Eq. (33), we have

$$r_\gamma(S) = \frac{\Gamma(\Upsilon(S) \to \chi_{b2}(1P) + \gamma)}{\Gamma(\Upsilon(S) \to \chi_{b0}(1P) + \gamma)} = \left(\frac{E_{\gamma 2}}{E_{\gamma 0}}\right)^3,$$

$$r_\gamma(D) = \frac{\Gamma(\Upsilon_1(D) \to \chi_{b2}(1P) + \gamma)}{\Gamma(\Upsilon_1(D) \to \chi_{b0}(1P) + \gamma)} = \left(\frac{E_{\gamma 2}}{E_{\gamma 0}}\right)^3,$$

where $E_{\gamma 2}$ and $E_{\gamma 0}$ respectively represent the photon energy of $V \to \chi_{b2}+\gamma$ and $V \to \chi_{b0}+\gamma$ (V stands for the initial vector state). From PDG data [66], we have $r_\gamma(2S) = 1.91 \pm 0.29$ and $r_\gamma(3S) = 3.82 \pm 1.05$, and the theoretical predictions of Eq. (36) and Eq. (37) are $r_\gamma(2S) = 1.57$, $r_\gamma(3S) = 3.6$, $r_\gamma(1D) = 0.0157$, and $r_\gamma(2D) = 0.036$.

So it is reasonable to treat $\Upsilon(2S)$ and $\Upsilon(3S)$ as pure S-wave states, and our conclusion of small S-D mixing angle is consistent with experiment for vector bottomonia below threshold.

If all observed vector bottomonia are S-wave dominant, the small Γ_{ee} of $\Upsilon(10580)$ and $\Upsilon(11020)$ naturally requires a large mixing angle under the S-D mixing mechanism, causing a large suppression of $r_\gamma(S)$. On the contrary, the B meson continuum suppresses $P_{b\bar{b}}$, leaving the ratio r_γ unchanged. Given that the deduced S-D mixing angle is small, we expect a large r_γ. Unfortunately, data on the radiative decay widths of $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ are still not available so far. A precise measurement of radiative decays will definitely tell us more about their internal structures.

For states above threshold, the predicted spectra and decay widths do not agree very well with experimental data. There are two reasons for this issue. As with most work, the meson loops of excited B mesons are ignored; however, this assumption may not be appropriate for highly excited states. For example, $\Upsilon(11020)$ is already 20 MeV heavier than the $B_1\bar{B}$ threshold. The second reason comes from the non-relativistic approximation of our bare mass. In principle, relativistic corrections become more important as the binding energy increases, so both the wave functions and the bare masses will change accordingly. However, including contributions of excited B mesons and refitting the spectrum and decay widths involves much more work, which lies beyond the scope of this paper. It remains a challenge to reproduce the spectra and dielectric or hadronic decay patterns.

5 Summary

In this paper, we make a thorough and precise calculation of coupled-channel effects in the framework of the 3P0 model with GEM for bottomonium. The results for the spectrum, open-channel strong decays, probabilities of the B meson continuum, S-D mixing, and vector meson dielectric decays are explicitly shown. To study near-threshold effects, we also plot the mass dependence of the mass shift and open-channel decay widths for pure S- and D-wave vector mesons.

For $\Upsilon(4S)$, the decay width from GEM can be two times as large as that from SHO over a wide energy range, and the mass shift is around three times as large. These large deviations indicate that SHO is not a good approximation for near-threshold states, even though the oscillator parameters are carefully selected to reproduce the root-mean-square radius of the corresponding mesons.

With the consideration of coupled-channel effects, we obtain small S–D mixing angles except for $\Upsilon(4S)$. We point out that for S-wave-dominated vector states, S–D mixing is not the only mechanism to suppress their dielectric decay widths; the B meson continuum can also lead to suppression. With this $B\bar{B}$ suppression mechanism at hand, we still succeed in reproducing the dielectric decays of vector bottomonia except for $\Upsilon(11020)$. The deviation between our predictions and experimental measurements for the spectrum and decays may be due to the neglect of excited B meson continuum in coupled-channel effects or the non-relativistic approximation in the quenched limit.

S–D mixing will cause suppression of the ratio in Eq. (36) for S-wave-dominated states; in contrast, the B meson continuum does not change this ratio. We suggest BaBar and Belle perform precise measurements of the radiative decays of $\Upsilon(10580)$, $\Upsilon(10860)$, and $\Upsilon(11020)$ to distinguish these two effects.

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