Fluid/gravity correspondence: Second order transport coefficients in compactified D4-branes (postprint)

Authors: Chao Wu, Yidian Chen, Mei Huang

Date: 2016-09-02T00:00:00+00:00

Abstract

We develop the boundary derivative expansion (BDE) formalism of fluid/gravity correspondence to nonconformal version through the compactified, near-extremal black D4-brane. We offer an explicit calculation of 9 second order transport coefficients, i.e., the , ∗, Π, 1,2,3 and 1,2,3 for the strongly coupled, uncharged and nonconformal relativistic fluid which is the holographic dual of compactified, near extremal black D4-brane. We also show that the nonconformal fluid considered in this work is free of causal problem and admits the relation 4 1− 2=2 .

Full Text

Preamble

Prepared for submission to JHEP

Fluid/gravity correspondence: Second order transport coefficients in compactified D4-branes

Chao Wu,^a Yidian Chen,^a Mei Huang^a,b

^aInstitute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, P.R. China
^bTheoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, P.R. China

Abstract: We develop the boundary derivative expansion (BDE) formalism of fluid/gravity correspondence to a nonconformal version through the compactified, near-extremal black D4-brane. We present an explicit calculation of 9 second-order transport coefficients—namely , ∗, Π, 1,2,3 and 1,2,3—for the strongly coupled, uncharged, and nonconformal relativistic fluid that is the holographic dual of the compactified, near-extremal black D4-brane. We also show that the nonconformal fluid considered in this work is free of causal problems and admits the relation 4 1 − 2 = 2 .

Keywords: Fluid/gravity correspondence, nonconformal relativistic fluid, second-order dynamical transport coefficients

---

Introduction

Relativistic hydrodynamics is an effective theory that successfully describes the dynamics of large numbers of classical or quantum particles in the long-wavelength, low-frequency limit at nonzero temperature and/or chemical potential. It has been applied with great success to phenomena across a wide range of areas, including high-energy nuclear collisions, astrophysics, and cosmology [?, ?].

Fluid dynamics is governed by the conservation of energy, momentum, and net charge number, with the equations of motion given by the conservation equations for the energy-momentum tensor Tμ and vector currents Jμ. To uniquely solve these partial differential equations, one must specify appropriate initial conditions. In the first-order hydrodynamical theories developed by Eckart [?] and Landau [?], the conserved quantities are expanded in terms of macroscopic degrees of freedom—local energy density , pressure p, charge densities na, four-velocity uμ, metric gμ (in curved spacetime), and their gradients. However, as pointed out in Ref.~[?], Eckart' s theory suffers from a severe problem: it permits infinite propagation speed for heat transfer, violating Einstein' s principle of relativity. Building on earlier work [?, ?], Ref.~[?] attempted to remedy this by generalizing the heat conduction equations and redefining the heat flow four-vector. Although Kranyš made further progress, the story was not yet complete. Müller [?] and later Israel and Stewart [?, ?, ?] identified that Kranyš' s work was problematic because it only considered first-order viscous terms in the entropy flux expression, which could be corrected by adding second-order viscous terms. Subsequent investigations [?, ?] demonstrated that the Müller-Israel-Stewart type second-order theory is the correct framework for relativistic dissipative hydrodynamics.

The primary driving force behind the development of relativistic hydrodynamics has been experiments at the Relativistic Heavy Ion Collider (RHIC), with Muronga being the first to apply second-order relativistic hydrodynamics to RHIC physics [?, ?, ?, ?]. Strictly speaking, the Müller-Israel-Stewart theory is not the complete second-order theory, prompting theorists to search for the correct and complete second-order formulation from both weak-coupling [?, ?, ?, ?, ?] and strong-coupling regimes [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?].

Before reviewing these literatures, we provide some general background on second-order relativistic fluids. In second-order theories of dissipative fluids, spacetime evolution of thermodynamic quantities depends not only on the equation of state but also on dissipative, non-equilibrium processes. Consequently, the conserved energy-momentum tensor must be expanded to include dissipative quantities such as viscosity, thermal conductivity, diffusion, and relaxation coefficients. Second-order theories have a hyperbolic structure that leads to well-posed initial-value (Cauchy) problems and causal propagation. The distinguishing feature of second-order theories is the relaxation time, which allows us to study the evolution of dissipative fluxes. For an uncharged nonconformal relativistic fluid in the Landau frame, the most general constitutive relation can be written as:

$$\begin{aligned}
T_{\mu\nu} &= pP_{\mu\nu} + \epsilon u_\mu u_\nu - 2\eta\sigma_{\mu\nu} - \zeta P_{\mu\nu}\theta \\
&+ 2\eta\tau_\pi \left\langle D\sigma_{\mu\nu} + \frac{1}{3}\sigma_{\mu\nu}\theta \right\rangle + 2\eta\tau_\pi^* \left\langle \sigma^\rho_{\mu\nu}\nabla_\rho \ln s \right\rangle \\
&+ \kappa \left\langle 2u^\rho u^\sigma R_{\rho\mu\sigma\nu} \right\rangle + \kappa^* \left\langle 2u^\rho u^\sigma R_{\rho\sigma} \right\rangle \\
&+ 4\lambda_1 \sigma_{\rho\mu}\sigma^\rho{}_\nu + 2\lambda_2 \sigma_{\rho\mu}\Omega^\rho{}_\nu + \lambda_3 \Omega_{\rho\mu}\Omega^\rho{}_\nu \\
&+ P_{\mu\nu}\left(\zeta\tau_\Pi D\theta + 4\xi_1 \sigma_{\rho\lambda}\sigma^{\rho\lambda} + \xi_2\theta^2 + \xi_3\Omega_{\rho\lambda}\Omega^{\rho\lambda} + \xi_4 P^{\rho\lambda}\nabla_\rho\nabla_\lambda \ln s \right. \\
&\left. + \xi_5 R + \xi_6 u^\rho u^\lambda R_{\rho\lambda}\right) + \lambda_4 \nabla_{\langle\mu} \ln s \nabla_{\nu\rangle} \ln s.
\end{aligned}$$

Here $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the spatial projection tensor, $\theta = \nabla_\rho u^\rho$ is the expansion term, and $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ are the Ricci and Riemann tensors associated with the metric $g_{\mu\nu}$. We adopt the same nomenclature for second-order transport coefficients as in Ref.~[**?**], which provides a standard prescription for constructing the energy-momentum tensor for uncharged relativistic fluids. The only difference from Ref.~[**?**] is that our shear tensor $\sigma_{\mu\nu}$ is half of theirs: $\sigma_{\mu\nu} = P^\rho{}_\mu P^\sigma{}_\nu \nabla_{(\rho} u_{\sigma)}$. To compensate, we include an additional factor of 2 in all viscous terms containing the shear tensor, which is why the viscous terms in Eq.~(1.1) have extra factors of 2 or 4 compared to Ref.~[**?**]. We also define the temporal or comoving derivative $D = u^\mu \nabla_\mu$ and the spatial-projected traceless symmetrized tensor, e.g., $\nabla_{(\rho} u_{\sigma)} = \frac{1}{2}(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho) - \frac{1}{3}P_{\rho\sigma}\nabla_\lambda u^\lambda$. With this definition, the shear viscous tensor is automatically spatial-projected and traceless: $\sigma_{\mu\nu} = \sigma_{\langle\mu\nu\rangle}$. The vorticity tensor is defined as $\Omega_{\mu\nu} = P^\rho{}_\mu P^\sigma{}_\nu \nabla_{[\rho} u_{\sigma]} = \frac{1}{2}P^\rho{}_\mu P^\sigma{}_\nu(\nabla_\rho u_\sigma - \nabla_\sigma u_\rho)$.

From Eq.~(1.1), we see that describing the dissipative properties of an uncharged nonconformal relativistic fluid up to second order requires 2 + 15 transport coefficients. Among these, 2 are first-order (η and ζ) and 15 are second-order. However, if the fluid is conformal and uncharged, only $1 + 5$ coefficients remain: η at first order and τ_π, κ, $\lambda_{1,2,3}$ at second order. The coefficients κ, κ^*, $\xi_{5,6}$ are related to curved spacetime and vanish for fluids in Minkowski space, which is appropriate for hot dense plasma in heavy ion collisions. Similarly, λ_4 and ξ_4 should not appear, according to constraints on second-order thermodynamic transport coefficients [?].

For the weak-coupling literature mentioned above, Refs.~[?, ?, ?] work in the dilute gas limit using kinetic theory via Grad' s moment expansion [?], while Refs.~[?, ?] employ conventional linear response theory where transport coefficients are calculated through Kubo formulas [?, ?] (for a modern pedagogical treatment, see Ref.~[?]). All strong-coupling works listed above use holography. Ref.~[?] directly calculates second-order (in derivative expansion) 2-point correlated transport coefficients κ and τ_π for $\mathcal{N}=4$ SYM plasma via the Green-Kubo formalism [?, ?, ?] of fluid/gravity duality. Recognizing that the original Green-Kubo formalism only handles 2-point correlators, Ref.~[?] generalized it to 3-point correlators, enabling Ref.~[?] to directly calculate second-order, 3-point correlated transport coefficients $\lambda_{1,2,3}$.

Refs.~[?, ?, ?, ?, ?, ?, ?, ?, ?] study second-order transport coefficients for various fluid systems within the BDE formalism of fluid/gravity correspondence. Refs.~[?, ?, ?, ?, ?] investigate 4D relativistic fluids using asymptotic AdS_5 backgrounds. Among these, [?, ?] establish the BDE formalism and study second-order coefficients [?] and entropy flux [?] for uncharged conformal fluids. Since Ref.~[?] is particularly representative, we record its result here in our conventions to help readers understand our work better:

$$T_{\mu\nu} = \frac{r_H^4}{2\kappa_5^2}\Big[P_{\mu\nu} + 3u_\mu u_\nu - \frac{2\eta}{r_H^3}\sigma_{\mu\nu} + \frac{r_H}{2}\big(\eta\tau_\pi\langle D\sigma_{\mu\nu}\rangle + \kappa\langle 2u^\rho u^\sigma R_{\rho\mu\sigma\nu}\rangle \\ + \lambda_1\sigma_{\rho\mu}\sigma^\rho{}_\nu + \lambda_2\sigma_{\rho\mu}\Omega^\rho{}_\nu + \lambda_3\Omega_{\rho\mu}\Omega^\rho{}_\nu\big)\Big],$$

with

$$\eta = \frac{r_H^3}{2\kappa_5^2}, \quad \eta\tau_\pi = \frac{r_H^2}{4\pi}, \quad \kappa = \frac{r_H^2}{8\pi}, \quad \lambda_1 = \frac{1}{8\pi}, \quad \lambda_2 = -\frac{1}{8\pi}\ln 2, \quad \lambda_3 = 0.$$

We have also included $\kappa = r_H^2/(8\pi)$ from Ref.~[?], obtained by directly calculating the 2-point Green-Kubo formula in AdS_5 black hole background. This result was also derived in the Weyl-covariant formulation of the BDE formalism [?]. Thus, the above equation provides a complete summary of the constitutive relation for strongly coupled uncharged SYM plasma corresponding to the AdS_5 black hole (with $2\kappa_5^2 = 1$). Note that $4\lambda_1 - \lambda_2 = 2\eta\tau_\pi$, a relation we will discuss in detail later.

Variations of this system, such as including a dilaton-dependent forcing term [?] or a U(1) conserved charge [?, ?], have also been investigated. Generalizations to different dimensional asymptotic AdS spacetimes include Ref.~[?] in AdS_4 and Refs.~[?, ?, ?] in AdS_{d+1}. Among these, Ref.~[?] adds matter fields to the

AdS_{d+1} black hole background of [?], while Ref.~[?] generalizes to curved boundaries. λ corrections to second-order transport coefficients of SYM plasma are studied in Refs.~[?, ?, ?, ?], and fluids corresponding to Gauss-Bonnet theories are examined in Refs.~[?, ?, ?].

It is also worthwhile to discuss classifications and constraints for the $2 + 15$ transport coefficients. These can be categorized as follows [?]: (1) From perturbative field theory, η, ζ, τ_π, τ_Π, κ, κ^*, and $\xi_{5,6}$ can be calculated from 2-point correlation functions in the Green-Kubo formalism, while τ_π^*, $\lambda_{1,2,3,4}$, and $\xi_{1,2,3,4}$ come from 3-point correlation functions. In an effective action formalism, the former are related to "linear" terms and the latter to "nonlinear" terms in the effective Lagrangian. (2) Regarding spacetime curvature, only κ, κ^*, and $\xi_{5,6}$ are related to curved metric. (3) From conformality, only η, τ_π, κ, and $\lambda_{1,2,3}$ appear in conformal fluids; any other coefficients indicate nonconformal behavior. (4) From the thermodynamic vs. dynamic perspective, κ, κ^*, $\lambda_{3,4}$, and $\xi_{3,4,5,6}$ are thermodynamic coefficients, while η, ζ, τ_π, τ_π^*, τ_Π, $\lambda_{1,2}$, and $\xi_{1,2}$ are dynamic. The reason for this distinction can be found in Ref.~[?].

The above remarks come from Ref.~[?], to which we add one observation: (5) Ref.~[?] shows that the 8 thermodynamic coefficients (κ, κ^*, $\lambda_{3,4}$, $\xi_{3,4,5,6}$) are constrained by positivity of the divergence of entropy flux, with 5 constraints, while the dynamic sector is free from these constraints. Thus, the number of independent second-order coefficients for any nonconformal uncharged relativistic fluid is 10. But why 5 constraints? A possible explanation is that all dynamic coefficients are related to either the shear tensor $\sigma_{\mu\nu}$ or expansion θ, so the corresponding viscous tensors can all increase entropy. The thermodynamic viscous tensors capture curvature and the inner heat flow $P^\nu_{\ \mu}\nabla_\nu \ln T$ of the fluid. Only viscous tensors involved in heat flux can cause entropy growth. Since the number of independent viscous tensors that the heat flux can compose is 3: $\nabla_\nu \ln T$, $P^\mu_{\ \nu}\sigma^{\nu\rho}\nabla_\rho \ln T$, and $P^\mu_{\ \nu}\Omega^{\nu\rho}\nabla_\rho \ln T$, the number of constraints should be $8 - 3 = 5$.

The references on second-order transport coefficients of strongly coupled relativistic fluids discussed above all concern conformal situations. Kanitscheider et al.~[?] studied first-order nonconformal hydrodynamics in Dp-branes using the BDE formalism in Fefferman-Graham coordinates [?], predicting a rough form for the second-order energy-momentum tensor but without explicit analytical results for second-order transport coefficients. Using this method, Ref.~[?] provided the first analytic second-order transport coefficients for nonconformal relativistic fluid corresponding to a scalar-deformed AdS_5 black hole background. The first numerical calculation of second-order transport coefficients for nonconformal fluid was performed in Ref.~[?], based on an Einstein+Scalar bottom-up holographic model. The authors numerically plotted the temperature-dependent behavior of τ_π, κ, κ^*, $\lambda_{3,4}$, and $\xi_{3,4,5,6}$ using Kubo relations from Ref.~[?] and the 5 constraints from Ref.~[?]. This numerical result represents the first step toward nontrivial temperature dependence for second-order transport properties of nonconformal fluid in the strong-coupling regime, offering crossover information for quark-gluon plasma (QGP).

Although Refs.~[?, ?] cover all second-order transport coefficients for uncharged nonconformal relativistic fluid, they both employ bottom-up holographic models. To better understand relativistic fluids from top-down models, further work is needed. With this goal, and building on our previous work [?] where we generalized the BDE formalism of fluid/gravity correspondence [?] to compactified, near-extremal black D4-branes at first order, we now proceed to second order in the same background to calculate transport coefficients. Through Ref.~[?] and the present work, we aim to provide a nonconformal counterpart to Bhattacharyya et al.' s AdS_5 construction [?] and improve our understanding of second-order transport properties for nonconformal relativistic fluids.

This paper is organized into 7 sections. Section 1 provides background and highlights our motivations. Section 2 briefly reviews the fluid/gravity correspondence techniques and first-order results for nonconformal fluid. Section 3 covers preliminaries for the second-order calculation. Section 4 addresses second-order constraint equations from the boundary fluid viewpoint, which will be helpful when investigating dynamical equations in Section 5 and expressing the constitutive relation in Section 6. Using the second-order metric perturbations solved in Section 5, we calculate the boundary stress tensor in Section 6 and discuss the final results in Section 7.

---

2 Brief Review of the First Order

In this section, we briefly review our framework at first order to prepare for the second-order calculation. Readers seeking more detail should consult Ref.~[?], where we developed a nonconformal version of fluid/gravity correspondence using compactified, near-extremal black D4-branes. The action for this system is

$$S = S_{\text{bulk}} - \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\,(2K + \mathcal{L}_{\text{ct}})\,,$$

where the second term is the Gibbons-Hawking term and the third is the counterterm, K is the trace of the extrinsic curvature, and the bulk action is

$$S_{\text{bulk}} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{10}{3}(\partial A)^2 - \frac{5}{3}(\partial B)^2 - V(\phi, A, B)\right],$$

with

$$V(\phi, A, B) = \frac{1}{2}e^{\frac{5A+8B}{3}} - e^{\frac{5A+2B}{3}}.$$

The equations of motion are

$$E_{MN} - T_{MN} = 0, \quad \nabla^2 \phi - \frac{\partial V}{\partial \phi} = 0, \quad \nabla^2 A - \frac{1}{10}\frac{\partial V}{\partial A} = 0, \quad \nabla^2 B - \frac{1}{5}\frac{\partial V}{\partial B} = 0,$$

where $E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R$ is the 5D Einstein tensor and

$$T_{MN} \equiv \frac{1}{2}\partial_M \phi \partial_N \phi + \frac{10}{3}\partial_M A \partial_N A + \frac{5}{3}\partial_M B \partial_N B - g_{MN}\left[\frac{1}{4}(\partial\phi)^2 + \frac{5}{3}(\partial A)^2 + \frac{5}{6}(\partial B)^2 + V\right]$$

is the bulk energy-momentum tensor. These equations are solved by the metric and scalar profiles

$$ds^2 = -r^{3/2} f(r) dt^2 + r^{3/2} d\vec{x}^2 + \frac{dr^2}{r^{3/2} f(r)}, \quad e^{\phi} = r^{3/4}, \quad e^A = r^{3/20}, \quad e^B = r^{3/5},$$

with

$$f(r) = 1 - \frac{r_H^3}{r^3},$$

which is reduced from the compactified near-extremal black D4-brane background. The metric is 5D asymptotically flat with a curvature singularity at $r = 0$. The Hawking temperature is $T = 3 r_H^{1/2}/(4\pi)$, which also gives the thermal equilibrium temperature of the system.

Re-expressing the metric in Eddington-Finkelstein coordinates $dv = dt + dr/(r^{3/2} f(r))$ and boosting the coordinates as $x^\mu \to x^\mu + u^\mu dx^\mu$, we obtain

$$ds^2 = -r^{3/2} f(r) u_\mu u_\nu dx^\mu dx^\nu + r^{3/2} P_{\mu\nu} dx^\mu dx^\nu - 2 r^{3/2} u_\mu dx^\mu dr,$$

with $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$, $u_\mu = \gamma(1, \beta_i)$, and $\gamma = 1/\sqrt{1-\beta_i^2}$. One can verify that both forms satisfy the Einstein equations given that $u^\mu u_\mu = -1$.

In the BDE formalism of fluid/gravity correspondence [**?**], r_H and u^μ in the metric are promoted to x^μ-dependent functions $r_H \to r_H(x^\mu)$ and $u^\mu \to u^\mu(x^\nu)$, called collective modes. They capture deviations from thermal equilibrium in the bulk metric. Note that the boundary where the fluid lives is flat, as in the original BDE formulation [**?**]. Therefore, second-order viscous terms related to κ, κ^*, and $\xi_{5,6}$ should not appear, as these coefficients only emerge when the fluid resides in curved spacetime. Similarly, λ_4 and ξ_4 should not be present, according to constraints on second-order thermodynamic transport coefficients [**?**]. Thus, we can preliminarily identify the potential candidates among the 15 second-order coefficients that our work may determine: τ_π, τ_π^*, τ_Π, $\lambda_{1,2,3}$, and $\xi_{1,2,3}$. It turns out that λ_3 and ξ_3 are trivial, similar to the case of λ_3 in $\mathcal{N}=4$ SYM plasma [**?**, **?**].

We elaborate further on these 9 candidates. Since τ_π and τ_Π are relaxation times due to shear and expansion dissipation, respectively, they should appear in this paper. τ_π^* indicates entry into the nonconformal regime associated with τ_π and should also be present. Regarding $\lambda_{1,2,3}$ and $\xi_{1,2,3}$, the only certainty is that ξ_2 should appear since it is associated with θ^2. For the rest, we can only state that if any (or both) of $\xi_{1,3}$ appear, the corresponding $\lambda_{1,3}$ must also appear, as viscous terms related to $\xi_{1,3,4}$ are the "trace" parts of those related to $\lambda_{1,3,4}$.

The subsequent steps are standard: (1) expand the boundary-dependent metric with respect to derivatives of collective modes; (2) add perturbations to the expanded metric and solve them from the 5D bulk equations of motion; (3) calculate the boundary stress tensor using the full 5D bulk metric with all perturbations present to extract the transport coefficients.

The full metric with first-order perturbations from [**?**] is

$$ds^2 = -r^{3/2} f(r_H(x), r) u_\mu u_\nu dx^\mu dx^\nu + r^{3/2}\left[P_{\mu\nu} + F(r_H(x), r)\sigma_{\mu\nu}\right] dx^\mu dx^\nu - 2r^{3/2}\left[1 + F_j(r_H(x), r)\partial_\rho u^\rho\right] u_\mu dx^\mu d$$

where $F(r)$, $F_j(r)$, and $F_k(r)$ are solved from first-order dynamical equations for metric perturbations:

$$F(r) = \frac{3\sqrt{r_H}}{2}\left[\arctan\left(\frac{\sqrt{r}}{\sqrt{r_H}}\right) - \frac{\sqrt{r_H}(\sqrt{r}-\sqrt{r_H})}{r+\sqrt{r r_H}+r_H}\right],$$

$$F_j(r) = \frac{F(r)}{3}, \quad F_k(r) = \frac{(r^3 + 2r_H^3)F(r)}{3r^3}.$$

Note that notation like $F(r)$ indicates r_H is x-independent, while $F(r_H(x), r)$ indicates x-dependence. The boundary stress tensor for the 5D theory is defined as

$$T_{\mu\nu} = \frac{1}{2\kappa_5^2}\left[K_{\mu\nu} - h_{\mu\nu}K + \frac{3}{\ell_{\mathrm{AdS}}}h_{\mu\nu}\right],$$

from which the boundary stress tensor for the relativistic fluid with first-order viscous terms is

$$T_{\mu\nu}^{(0)+(1)} = \frac{r_H^3}{2\kappa_5^2}\left[P_{\mu\nu} + 3u_\mu u_\nu - \frac{2\eta}{r_H^3}\sigma_{\mu\nu} + \frac{\zeta}{r_H^3}\partial_\rho u^\rho P_{\mu\nu}\right].$$

In the above and throughout this paper, we set $2\kappa_5^2 = 1$ and will restore it at the end. $T^{(0)}_{\mu\nu}$ is the ideal fluid energy-momentum tensor containing only the first two terms on the right-hand side of Eq.~(2.15).

---

3 Setup of the Second Order Calculation

At each order in the BDE formalism, the first step is to obtain the correctly expanded bulk metric. The second-order calculation is significantly more complicated than the first order. In this section, we provide a detailed derivation of the second-order expanded metric.

We begin by expanding $r_H(x)$ and $\beta_i(x)$ to second order in Eq.~(2.12):

$$r_H(x) = r_H+\delta r_H+\delta^2 r_H+\delta r_H^{(1)}, \quad \beta_i(x) = \delta\beta_i+\delta^2\beta_i, \quad u^\mu(x) = \left(1+\frac{\delta\beta_i\delta\beta_i}{2}, \delta\beta_i+\delta^2\beta_i\right),$$

where we denote $r_H \equiv r_H^{(0)}$ and $\delta\#, \delta^2\#$ are shorthand for $x^\mu\partial_\mu\#$ and $x^\mu x^\nu\partial_\mu\partial_\nu\#$, respectively. $r_H^{(1)}$ is the first-order collective mode for the boundary relativistic fluid, independent of the first-order source $x^\mu\partial_\mu r_H^{(0)}$ that comes from derivative expansion of $r_H^{(0)}$. Then $f(r_H(x), r)$, $F(r_H(x), r)$, $F_j(r_H(x), r)$, and $F_k(r_H(x), r)$ in Eq.~(2.12) can be expanded as

$$\begin{aligned}
f(r_H(x), r) &= f(r) - \frac{3r_H^2\delta r_H}{r^3} - \frac{3r_H^2\delta^2 r_H}{r^3} + \frac{6r_H(\delta r_H)^2}{r^3} + \frac{3r_H^2\delta r_H^{(1)}}{r^3},\\
F(r_H(x), r) &= F(r) + F'(r)\delta r_H + F'(r)\delta^2 r_H + \frac{1}{2}F''(r)(\delta r_H)^2 + F'(r)\delta r_H^{(1)},\\
F_j(r_H(x), r) &= F_j(r) + F_j'(r)\delta r_H + F_j'(r)\delta^2 r_H + \frac{1}{2}F_j''(r)(\delta r_H)^2 + F_j'(r)\delta r_H^{(1)},\\
F_k(r_H(x), r) &= F_k(r) + F_k'(r)\delta r_H + F_k'(r)\delta^2 r_H + \frac{1}{2}F_k''(r)(\delta r_H)^2 + F_k'(r)\delta r_H^{(1)},
\end{aligned}$$

where $f(r)$, $F(r)$, $F_j(r)$, and $F_k(r)$ denote functions with r_H independent of x. For any of them, e.g., $F(r)$, we simply denote it as F and its derivatives as F', F'', etc., to simplify notation.

Thus, Eq.~(2.12) can be expanded to second order in boundary derivatives as

$$\begin{aligned}
ds^2 = &-r^{3/2}f(r)u_\mu u_\nu dx^\mu dx^\nu + r^{3/2}P_{\mu\nu}dx^\mu dx^\nu - 2r^{3/2}u_\mu dx^\mu dr \\
&+ r^{3/2}\left[F_k(r)\partial_\rho u^\rho + F_k'(r)\delta r_H \partial_\rho u^\rho + \frac{1}{2}F_k''(r)(\delta r_H)^2\partial_\rho u^\rho + F_k'(r)\delta r_H^{(1)}\partial_\rho u^\rho\right]u_\mu u_\nu dx^\mu dx^\nu \\
&+ r^{3/2}\left[F(r)\sigma_{\mu\nu} + F'(r)\delta r_H \sigma_{\mu\nu} + \frac{1}{2}F''(r)(\delta r_H)^2\sigma_{\mu\nu} + F'(r)\delta r_H^{(1)}\sigma_{\mu\nu}\right]dx^\mu dx^\nu \\
&- 2r^{3/2}\left[F_j(r)\partial_\rho u^\rho + F_j'(r)\delta r_H \partial_\rho u^\rho + \frac{1}{2}F_j''(r)(\delta r_H)^2\partial_\rho u^\rho + F_j'(r)\delta r_H^{(1)}\partial_\rho u^\rho\right]u_\mu dx^\mu dr \\
&+ \text{terms from expanding } u^\mu(x) \text{ and } r_H(x).
\end{aligned}$$

The complete expanded form is given in Eq.~(3.3) of the original text, where $\partial_\beta \equiv \partial_i\beta_i$, $\partial_0 \equiv \partial_v$, and $\delta\beta_{(i}\partial_{\beta_{j)}} = \frac{1}{2}(\delta\beta_i\partial_0\beta_j + \delta\beta_j\partial_0\beta_i)$.

---

4 The Second Order Constraints and Navier-Stokes Equations

In this section, we discuss constraint equations and Navier-Stokes equations at second order. The constraint equation is

$$\partial_\mu\partial_\rho T^{(0)\rho}_{\ \nu} = 0,$$

and the Navier-Stokes equation is

$$\partial_\mu T^{(0+1)\mu}_{\ \nu} = 0.$$

From Eq.~(4.1), setting μ, ν to 0 or i, we derive constraint relations satisfied by second-order spatial viscous terms:

$$\begin{aligned}
v_{1i} + v_{4i} + 2S_1 + v_{1i} + 2v_{2i} - v_{5i} - S_3 &= 0, \\
S_4 + S_5 &= 0, \\
V_{1i} - V_{2i} + 2V_{3i} &= 0, \\
V_{2i} - 4V_{1i} - V_{4i} - v_{3i} + T_{5ij} + 2T_{6ij} &= 0, \\
V_{3i} &= 0, \\
V_{5i} &= 0, \\
T_{4ij} + 4T_{1ij} + t_{1ij} + 2t_{2ij} - \frac{1}{3}\delta_{ij}S_4 + \frac{1}{3}\delta_{ij}S_5 + \frac{1}{3}\delta_{ij}S_1 + \frac{1}{3}\delta_{ij}S_2 &= 0.
\end{aligned}$$

The explicit forms of these terms are given in Table 1 . Here $l_i = \epsilon_{ijk}\partial_j\beta_k$ is the pseudo-vector associated with the vorticity tensor $\Omega_{\mu\nu}$, and $\sigma_{ij} = \partial_{(i}\beta_{j)} - \frac{1}{3}\delta_{ij}\partial_\beta$ are the spatial components of $\sigma_{\mu\nu}$.

It is necessary to explain the meaning of these constraints, Eqs.~(4.3)-(4.8). They arise from expanding all components of Eq.~(4.1) to second order: Eqs.~(4.3) and (4.4) are the (00) and (ii) components; (4.5) and (4.6) are the ($0i$) and ($i0$) components; the last two are the antisymmetric and symmetric parts of the traceless tensor sector, respectively.

The Navier-Stokes equations at second order are obtained by expanding Eq.~(4.2) to second order. For $\nu = 0$:

$$\partial_\mu T^{(0+1)\mu}{}_0 = \frac{3}{2} r_H^{1/2} \partial_0 r_H^{(1)} + (\text{terms involving } \partial r_H, \partial\beta) = 0.$$

One might expect an equation for $\partial_0 r_H^{(1)}$ with second-order viscous terms in the scalar sector, as in [**?**]. The offending x-dependent terms should vanish by themselves, which can be shown using first-order constraint equations [**?**]:

$$\partial_0 r_H = \partial_i r_H = -\frac{2}{3}\partial_0 \beta_i.$$

Then the x-dependent part can be re-expressed as

$$\frac{3}{2} r_H^{1/2}\left[\partial_0 r_H^{(1)} - \left(\frac{3}{2}\partial_0\partial_\beta + \frac{1}{4}(\partial_\beta)^2 + \frac{3}{2}\partial_0\beta_i\partial_0\beta_i\right)\right] = 0,$$

where the bracketed terms correspond to Eq.~(4.3) and Eq.~(4.6). Thus, the $\nu = 0$ component of Eq.~(4.2) finally yields one of the Navier-Stokes equations for nonconformal hydrodynamics at second order:

$$\partial_0 r_H^{(1)} = 0.$$

The $\nu = i$ component of Eq.~(4.2) gives

$$\partial_\mu T^{(0+1)\mu}{}_i = \frac{3}{2} r_H^{1/2} \partial_i r_H^{(1)} + (\text{terms involving } \partial r_H, \partial\beta) = 0,$$

where the x-dependent part can be rewritten using Eq.~(4.5) and Eq.~(4.8), yielding another Navier-Stokes equation at second order:

$$\partial_i r_H^{(1)} = 0.$$

Equations (4.12) and (4.15) can be derived as constraint equations from the bulk gravity theory, as we will show in the next section.

5 The Second Order Perturbations

In this section, we solve the second-order perturbations. Following the scheme of [?], we work in the gauge where $g_{rr} = 0$ and $g_{\mu r} \propto u_\mu$. This gauge offers the convenience that we need not consider fluctuations of the (rr) and (ir) components of the bulk metric. The perturbation ansatz at second order is:

$$g^{(2)}_{\mu\nu} = k^{(2)}(r)u_\mu u_\nu + 2w^{(2)}_{(\mu}(r)u_{\nu)} + r^{3/2}\left[h^{(2)}(r)\delta_{ij} + \alpha^{(2)}_{ij}(r)\right]dx^i dx^j.$$

All unknown functions have the form

$$F_I(r) \times (\text{2nd order viscous terms}),$$

where $F_I(r)$ are functions of r and the second-order viscous terms are listed in Table 1. Although $h^{(2)}$ and $\alpha^{(2)}_{ij}$ come from different sectors, we write them together to emphasize their similarities in solving the equations of motion, since h is related to the trace part of α_{ij}. For simplicity, we set $r_H = 1$ from now on and will restore it when presenting our final results.

5.1 The Tensor Part

We begin with the tensor part. The equation of motion for $\alpha^{(2)}_{ij}$ is

$$E_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}E_{kl} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl}.$$

Substituting the second-order expanded metric (3.3) yields the differential equation for $\alpha^{(2)}_{ij}$:

$$\frac{d}{dr}\left(r^4 f(r)\frac{d\alpha^{(2)}_{ij}}{dr}\right) = S^{(\alpha)}_{ij},$$

where the source term $S^{(\alpha)}_{ij}$ is given by a lengthy expression involving various combinations of F, F_j, F_k and their derivatives, multiplied by different viscous terms (t_{3ij}, T_{1ij}, T_{4ij}, T_{6ij}, T_{7ij}). The full expression is provided in Eq.~(5.4) of the original text.

This second-order differential equation is much more complex than its first-order counterpart and generally does not admit an analytical solution. Since we only care about its large-r behavior, we perform a large-r expansion during the solving process. Remarkably, the left-hand side of Eq.~(5.4) is the same as at first order (if multiplied by $2r$), manifesting key features of the BDE formalism: (1) the formalism is linear in the r direction but nonlinear in x^μ directions; (2) the homogeneous part of the differential equations in r is the same at every order, while the nonhomogeneous source terms differ order by order.

We write the formal solution of Eq.~(5.4) as

$$\alpha^{(2)}_{ij}(r) = \int_\infty^r \frac{dx}{x^4 f(x)} \int_{r_H}^x dy\, y\, S^{(\alpha)}_{ij}(y),$$

where the source term $S^{(\alpha)}_{ij}(y)$ has several independent branches, as each second-order viscous term can be treated independently. For example, we may solve Eq.~(5.4) with only t_{3ij} present first, then with only T_{4ij}, and so on. The final solution is obtained by summing all these "subsolutions."

Since we cannot integrate Eq.~(5.4) directly to obtain an analytic solution, we proceed as follows: (1) perform the inner integration directly; (2) expand the first integrated result at large r; (3) perform the outer integration. The final result, keeping terms up to $1/r^3$ (which contribute to the boundary stress tensor), is:

$$\alpha^{(2)}_{ij}(r) = \frac{1}{r^3}\left[\left(\frac{9\sqrt{3}-2\ln 3}{45\sqrt{3}}\right)(t_{3ij}+T_{1ij}) + \left(\frac{2\ln 3-9\sqrt{3}}{45\sqrt{3}}\right)T_{4ij} + \frac{4}{5}T_{6ij} + \left(\frac{2\ln 3-45\sqrt{3}}{45\sqrt{3}}\right)T_{7ij}\right].$$

Regarding integration constants: based on first-order experience, each branch has two constants—one fixed by regularity at $r = r_H$, the other by normalization at infinity. Additional constants are fixed by the Landau frame condition when expressing the boundary stress tensor. In general, we should write the solution as

$$\alpha^{(2)}_{ij}(r) = \int_\infty^r \frac{dx}{x^4 f(x)} \int_{r_H}^x dy\, y\, S^{(\alpha)}_{ij}(y) + C_1^{(I)} \int_\infty^r \frac{dx}{x^4 f(x)} + C_2^{(I)},$$

where I is summed over all viscous terms in $S^{(\alpha)}_{ij}$. Since Eq.~(5.6) is already regular at $r = r_H$ and asymptotically vanishes, all constants in the above equation should be zero.

5.2 The Vector Part

The constraint equation for the vector part is

$$g_{r0}(E_{0i} - T_{0i}) + g_{rr}(E_{ri} - T_{ri}) = 0.$$

Using the second-order expanded metric (3.3), this gives

$$\partial_i r_H^{(1)} + (\text{terms involving } v_{4i}, v_{5i}, V_{1i}, V_{2i}, V_{3i}) = 0.$$

Expanding to $1/r^3$ order yields the vector constraint equation at second order:

$$\partial_i r_H^{(1)} + \frac{1}{3}(v_{4i} + v_{5i} + V_{1i} - V_{2i} - V_{3i}) = 0,$$

which is precisely the second Navier-Stokes equation (4.15).

The dynamical equation of the vector sector is

$$E_{ri} - T_{ri} = 0,$$

which gives

$$\frac{d}{dr}\left(r^4 f(r)\frac{dw_i^{(2)}}{dr}\right) = S_i^{(w)},$$

where $S_i^{(w)}$ is the source term for vector perturbations. Three of its five branches contain the divergence part $4V_{1i} + 2V_{2i} + 4V_{3i}$, so $w_i^{(2)}(r)$ can be integrated as

$$w_i^{(2)}(r) = \int \frac{S_i^{(w)}(x)}{x^4 f(x)}dx.$$

The solution is

$$w_i^{(2)}(r) = \frac{1}{r^2}(4V_{1i} + 2V_{2i} + V_{3i}) + (\text{other terms involving } v_{4i}, v_{5i}, V_{1i}, V_{2i}, V_{3i}).$$

The term $r^{-2}(4V_{1i}+2V_{2i}+V_{3i})$ comes from the indefinite integral of the divergent part of the source term $S_i^{(w)}$. In fact, the vector perturbation does not contribute to the boundary stress tensor because it is trivial in our framework, similar to the case in Ref.~[**?**]. Perturbations only contribute to the boundary stress tensor if they contain terms of order $1/r^3$.

5.3 The Scalar Part

The scalar sector remains the most complicated part at second order, but we benefit greatly from our first-order experience.

The scalar part contributes to the stress tensor of the boundary fluid, so we must solve all three scalar perturbations explicitly.

The scalar constraint equations are:

$$g_{rr}(E_{r0} - T_{r0}) + g_{r0}(E_{00} - T_{00}) = 0,$$

$$g_{rr}(E_{rr}-T_{rr})+g_{r0}(E_{r0}-T_{r0})=0.$$

The first gives

$$\partial_0 r_H^{(1)}+(\text{terms involving } S_1,S_4,S_5)=0.$$

Expanding to order $1/r^4$ yields

$$\partial_0 r_H^{(1)}+\frac{1}{3}(2S_1+S_4+S_5)=0.$$

Note that the terms in the second bracket correspond to Eq.~(4.4) and thus equal zero. So we reproduce the first Navier-Stokes equation (4.12).

The second scalar constraint equation gives a relation between $h^{(2)}$, $j^{(2)}$, and $k^{(2)}$:

$$3(5r^3-2)h'^{(2)}-30r^2j^{(2)}-\frac{5}{2}F'k'^{(2)}=(\text{source terms}).$$

There are 7 dynamical equations in the scalar part, but only 3 are independent: the (rr) and (ii) components of the Einstein equation (2.3) and the scalar field equation (2.4). Following our first-order procedure, we choose the (rr) component of the Einstein equation

$$6rh''^{(2)}+9h'^{(2)}+10j'^{(2)}=(\text{source terms}),$$

the scalar field equation

$$r^3f(r)j'^{(2)}+6r^2j^{(2)}+k'^{(2)}-\frac{1}{2}r^3f(r)h'^{(2)}=(\text{source terms}),$$

together with the second scalar constraint (5.20) to solve the scalar perturbations.

First, we use Eqs.~(5.20), (5.21), and (5.22) to eliminate j and k, obtaining the differential equation for $h^{(2)}$:

$$\frac{d}{dr}\left(r^4f(r)\frac{dh^{(2)}}{dr}\right)=S_h=c_1^{(h)}(r)s_3+c_2^{(h)}(r)S_1+c_3^{(h)}(r)S_3+c_4^{(h)}(r)S_4+c_5^{(h)}(r)S_5,$$

where S_h is the source term and the coefficient functions $c^{(h)}(r)$ are given by expressions involving F, F_j, F_k and their derivatives (see Eqs.~(5.24) in the original text).

Thus, $h^{(2)}$ can be solved by

$$h^{(2)}(r) = \int_{\infty}^{r} \frac{dx}{x^4 f(x)} \int_{r_H}^{x} dy\, y\, S_h(y).$$

This integral is performed similarly to $\alpha_{ij}^{(2)}$, except that after the inner integration we must expand to at least order $1/r^6$. This is because the three scalar perturbations are mixed and have different asymptotic behaviors:

$$F(r) \sim \frac{1}{r^3}, \quad F_j(r) \sim \frac{1}{r^6}, \quad F_k(r) \sim \text{constant}.$$

The largest gap in asymptotic behavior is $1/r^3$ between F_j and F_k. Terms of order $1/r^6$ in F_j still affect terms of order $1/r^3$ in F_k when solving the scalar perturbation equations. Since terms of order $1/r^3$ contribute to the boundary stress tensor, we must solve $h^{(2)}$ and $j^{(2)}$ to order $1/r^6$ to obtain correct $1/r^3$ terms in $k^{(2)}$.

The integrated result for $h^{(2)}(r)$ is

$$h^{(2)}(r) = \frac{1}{r^6} \left[\left(\frac{405\sqrt{3} - 2\ln 3}{270\sqrt{3}} \right) s_3 + \left(\frac{9\sqrt{3} - 2\ln 3}{54\sqrt{3}} \right) S_1 + \left(\frac{45\sqrt{3} - 2\ln 3}{270\sqrt{3}} \right) S_3 + \left(\frac{45\sqrt{3} - 2\ln 3}{270\sqrt{3}} \right) S_4 + \left(\right.\right.$$

Here $h^{(2)}(r)$ has no nontrivial integration constants, as seen from the asymptotic behavior of its first-order counterpart $F(r)$ (5.26).

Regarding integration constants for the scalar sector: in our first-order solution [**?**], only one branch ($\partial_i \beta_i$) was present, with 4 integration constants—two for h and one each for j and k—because the differential equation for h is second-order in r while those for j and k are first-order. The situation is the same at second order, except there are now 5 branches: s_3, S_1, S_3, S_4, and S_5. Thus, the total number of integration constants should be $4 \times 5 = 20$, with 10 belonging to $h^{(2)}$ and 5 each to $j^{(2)}$ and $k^{(2)}$. From the solution for $h^{(2)}$, we see that all 10 integration constants must be zero.

The differential equation for $j^{(2)}$ is

$$j'^{(2)}(r) = S_j = c_1^{(j)}(r) s_3 + c_2^{(j)}(r) S_1 + c_3^{(j)}(r) S_3 + c_4^{(j)}(r) S_4 + c_5^{(j)}(r) S_5,$$

where the coefficient functions $c^{(j)}(r)$ are given in Eqs.~(5.31)-(5.32). The solution is

$$j^{(2)}(r) = \int_{\infty}^{r} S_j(x) dx,$$

which yields

$$j^{(2)}(r) = \frac{1}{r^6}\left[\left(\frac{36\sqrt{3}-2\ln 3}{270\sqrt{3}}\right)s_3 + \left(\frac{18\sqrt{3}-2\ln 3}{270\sqrt{3}}\right)S_1 + \left(\frac{270\sqrt{3}-2\ln 3}{270\sqrt{3}}\right)S_3 + \left(\frac{36\sqrt{3}-2\ln 3}{270\sqrt{3}}\right)S_4 + \right.$$

From the large-r behavior of F_j (5.27), we see that no integration constants are needed here either.

Substituting $h^{(2)}$ and $j^{(2)}$ into Eq.~(5.20) gives the differential equation for $k^{(2)}$:

$$k'^{(2)}(r) = S_k = c_1^{(k)}(r)s_3 + c_2^{(k)}(r)S_1 + c_3^{(k)}(r)S_3 + c_4^{(k)}(r)S_4 + c_5^{(k)}(r)S_5,$$

where the coefficient functions $c^{(k)}(r)$ are given in Eqs.~(5.35). The asymptotic behavior of S_k contains a divergent term $8rS_1$ in the S_1 branch, which contributes to the result as an indefinite integral $\int 8r\,dr = 4r^2$. Thus, $k^{(2)}$ can be solved by

$$k^{(2)}(r) = 4r^2S_1 - \int_\infty^r (S_k(x) - 8xS_1)dx + C_{k1}s_3 + C_{k2}S_1 + C_{k3}S_3 + C_{k4}S_4 + C_{k5}S_5,$$

where C_{k1} to C_{k5} are integration constants fixed by requiring the boundary stress tensor to be in the Landau frame:

$$C_{k1} = C_{k4} = \frac{90\sqrt{3}-2\ln 3}{675\sqrt{3}}, \quad C_{k2} = \frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}, \quad C_{k5} = \frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}, \quad C_{k3} = \frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}.$$

These constants are necessary because the lowest term in F_k starts from a constant (5.28), and one cannot obtain constant terms from only the integration part $\int_\infty^r (S_k(x) - 8xS_1)dx$ in Eq.~(5.37). Thus, we must add the C_{ki} "by hand."

The final result for $k^{(2)}$ is

$$k^{(2)}(r) = 4r^2S_1 + \frac{1}{r^3}\left[\left(\frac{90\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)s_3 + \left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)S_1 + \left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)S_3 + \left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)S\right.$$

Collecting all second-order perturbations solved in this section with Eq.~(3.3), we obtain the complete metric in global form up to second order:

$$ds^2 = -r^{3/2} f(r_H(x), r) u_\mu u_\nu dx^\mu dx^\nu + r^{3/2} \left[P_{\mu\nu} + h(x,r) P_{\mu\nu} + \alpha_{\mu\nu}(x,r) \right] dx^\mu dx^\nu - 2r^{3/2} \left[1 + j(x,r) \right] u_\mu dx^\mu dr + r$$

where $\#(x,r) = \#^{(1)}(x,r) + \#^{(2)}(x,r)$ for $\# = k, h, j, \alpha_{\mu\nu}$, and the first-order perturbations are

$$\begin{aligned} k^{(1)}(x,r) &= F_k(r_H(x), r) \partial_\rho u^\rho, \\ h^{(1)}(x,r) &= F(r_H(x), r) \partial_\rho u^\rho, \\ j^{(1)}(x,r) &= F_j(r_H(x), r) \partial_\rho u^\rho, \\ \alpha^{(1)}_{\mu\nu}(x,r) &= F(r_H(x), r) \sigma_{\mu\nu}. \end{aligned}$$

The second-order perturbations $\#^{(2)}(x,r)$ are given by the corresponding results solved in this section, and $w^{(2)}_\mu$ is given by Eq.~(5.15).

---

6 The Boundary Stress Tensor at Second Order

Similar to first order, the second-order boundary stress tensor contains tensor and scalar parts:

$$T^{(2)}_{\mu\nu} = \pi^{(2)}_{\mu\nu} + P_{\mu\nu} \Pi^{(2)}.$$

The tensor part extracted from the Brown-York energy-momentum tensor of the second-order full metric is

$$\pi^{(2)}_{ij} = \frac{r_H^3}{2\kappa_5^2} \frac{1}{r^3} \left[\left(\frac{9\sqrt{3} - 2\ln 3}{45\sqrt{3}} \right) (t_{3ij} + T_{1ij}) + \left(\frac{2\ln 3 - 9\sqrt{3}}{45\sqrt{3}} \right) T_{4ij} + \frac{4}{5} T_{6ij} + \left(\frac{2\ln 3 - 45\sqrt{3}}{45\sqrt{3}} \right) T_{7ij} \right],$$

and the scalar part is

$$\Pi^{(2)} = \frac{r_H^3}{2\kappa_5^2} \frac{1}{r^3} \left[\left(\frac{45\sqrt{3} - 2\ln 3}{675\sqrt{3}} \right) s_3 + \left(\frac{45\sqrt{3} - 2\ln 3}{675\sqrt{3}} \right) S_1 + \left(\frac{45\sqrt{3} - 2\ln 3}{675\sqrt{3}} \right) S_3 + \left(\frac{45\sqrt{3} - 2\ln 3}{675\sqrt{3}} \right) S_4 + \right.$$

To obtain the covariant form of the boundary stress tensor, we use the replacements

$$
\begin{aligned}
t_{3ij} &= \partial_0 \sigma_{ij} \to D\sigma_{\mu\nu}, \\
T_{1ij} &= \partial_v \beta_i \partial_v \beta_j - \frac{1}{3}\delta_{ij} S_1 \to \sigma_{\mu\rho}\sigma^{\rho}{}_{\nu}, \\
T_{4ij} &= \sigma_{ij}\partial_\beta \to \sigma_{\mu\nu}\partial_\rho u^\rho, \\
T_{6ij} &= \sigma_{ik}\sigma_{kj} - \frac{1}{3}\delta_{ij} S_5 \to \sigma_{\mu\rho}\sigma^{\rho}{}_{\nu} - \frac{1}{3}P_{\mu\nu}\sigma_{\rho\lambda}\sigma^{\rho\lambda}, \\
T_{7ij} &= 2\epsilon_{kl(i}\sigma_{j)l} l_k \to \sigma_{\mu\rho}\Omega^{\rho}{}_{\nu} + \sigma_{\nu\rho}\Omega^{\rho}{}_{\mu},
\end{aligned}
$$

for the tensor sector, and

$$
\begin{aligned}
s_3 &= \partial_i^2 r_H \to P^{\mu\nu}\partial_\mu\partial_\nu r_H, \\
S_1 &= \partial_0\beta_i\partial_0\beta_i \to Du^\mu Du_\mu, \\
S_3 &= (\partial_i\beta_i)^2 \to (\partial_\mu u^\mu)^2, \\
S_4 &= l_i l_i \to l^\mu l_\mu = 2\Omega_{\mu\nu}\Omega^{\mu\nu}, \\
S_5 &= \sigma_{ij}\sigma_{ij} \to \sigma_{\mu\nu}\sigma^{\mu\nu},
\end{aligned}
$$

for the scalar sector. The scalar sector then becomes

$$
\Pi^{(2)} = \frac{r_H^3}{2\kappa_5^2}\frac{1}{r^3}\left[\left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)P^{\mu\nu}\partial_\mu\partial_\nu r_H + \left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)(\partial_\mu u^\mu)^2 + \left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)2\Omega_{\mu\nu}\Omega^{\mu\nu} + \left(4\right.\right.
$$

To match the definition of the constitutive relation for nonconformal fluid in [?], we use the covariant form of Eq.~(4.4),

$$
D\partial_\mu u^\mu = P^{\mu\nu}\partial_\mu\partial_\nu r_H + \frac{3}{2}Du^\mu Du_\mu - \frac{1}{3}(\partial_\mu u^\mu)^2 + \Omega_{\mu\nu}\Omega^{\mu\nu},
$$

to re-express $\Pi^{(2)}$ as

$$
\Pi^{(2)} = \frac{r_H^3}{2\kappa_5^2}\frac{1}{r^3}\left[\left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)D\partial_\mu u^\mu + \left(\frac{225-675\sqrt{3}-6\ln 3}{675\sqrt{3}}\right)(\partial_\mu u^\mu)^2 + \left(\frac{45\sqrt{3}-2\ln 3}{675\sqrt{3}}\right)\sigma_{\mu\nu}\sigma^{\mu\nu}\right].
$$

Thus, the final form of the boundary stress tensor up to second-order derivative expansion is

$$
\begin{aligned}
T_{\mu\nu} = \frac{r_H^3}{2\kappa_5^2}\Big[&P_{\mu\nu} + 3u_\mu u_\nu - \frac{2\eta}{r_H^3}\sigma_{\mu\nu} + \frac{\zeta}{r_H^3}\partial_\rho u^\rho P_{\mu\nu} \\
&+ \frac{1}{r_H^3}\left(\eta\tau_\pi\langle D\sigma_{\mu\nu}\rangle + \eta\tau_\pi^*\langle\sigma_{\mu\nu}\partial_\rho u^\rho\rangle + \lambda_1\sigma_{\mu\rho}\sigma^\rho{}_\nu + \lambda_2\sigma_{\mu\rho}\Omega^\rho{}_\nu + \lambda_3\Omega_{\mu\rho}\Omega^\rho{}_\nu\right) \\
&+ P_{\mu\nu}\left(\zeta\tau_\Pi D\partial_\rho u^\rho + \xi_1\sigma_{\rho\lambda}\sigma^{\rho\lambda} + \xi_2(\partial_\rho u^\rho)^2 + \xi_3\Omega_{\rho\lambda}\Omega^{\rho\lambda}\right)\Big].
\end{aligned}
$$

Here we have restored r_H and κ_5. Comparing with the standard energy-momentum tensor of relativistic fluid, we can read off all transport coefficients:

$$
\begin{aligned}
\eta &= \frac{r_H^3}{2\kappa_5^2}, \quad \zeta = 0, \\
\eta\tau_\pi &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{5}{6\sqrt{3}} - \frac{\ln 3}{6}\right), \\
\eta\tau_\pi^* &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{5}{6\sqrt{3}} + \frac{\ln 3}{6}\right), \\
\lambda_1 &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{225 - 675\sqrt{3} - 6\ln 3}{675\sqrt{3}}\right), \\
\lambda_2 &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{45\sqrt{3} - 2\ln 3}{675\sqrt{3}}\right), \\
\lambda_3 &= 0, \\
\zeta\tau_\Pi &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{5}{6\sqrt{3}} - \frac{\ln 3}{6}\right), \\
\xi_1 &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{45\sqrt{3} - 2\ln 3}{675\sqrt{3}}\right), \\
\xi_2 &= \frac{r_H^2}{2\kappa_5^2}\left(\frac{225 - 675\sqrt{3} - 6\ln 3}{675\sqrt{3}}\right), \\
\xi_3 &= 0.
\end{aligned}
$$

The appearance of τ_π^*, τ_Π, and $\xi_{1,2}$ indicates that we are in the nonconformal regime. There are two simple relations among the second-order coefficients in Eq.~(6.10), given that $c_s^2 = 1/5$:

$$
\tau_\pi = \tau_\Pi, \quad \xi_1 = \frac{1}{3}(1 - 3c_s^2)\tau_\pi, \quad \xi_2 = \frac{1}{3}(1 - 3c_s^2)\tau_\pi.
$$

These relations match predictions made in [**?**] about nonconformal fluids [**?**]. However, some relations from [**?**] are not satisfied in our work, such as $\tau_\pi^* = (1 - 3c_s^2)\tau_\pi$. This may be due to different asymptotic behaviors of the bulk metric in the first 4 directions: in Ref.~[?] it is proportional to r^2, while in our case it is $r^{3/2}$. This changes the power law between thermal (and hydrodynamic) quantities such as entropy density and temperature. For example, the power law between entropy density and temperature is $s \propto T^5$ in our work. It is these deviations in power law that modify some nonconformal transport coefficients and violate certain relations predicted in [?].

Another important relation satisfied by the coefficients in Eq.~(6.10) is

$$4\lambda_1 - \lambda_2 = 2\eta\tau_\pi,$$

first found in Ref.~[?] for charged AdS$_5$ black hole systems and shown to hold for any chemical potential. The authors of [?] also pointed out that this relation is satisfied in asymptotic AdS black holes of any dimension [?]. Later, Ref.~[?] proved it holds for a large class of strongly coupled conformal plasmas with matter fields in any dimension. Further study [?] showed it remains valid under λ corrections. Even with Gauss-Bonnet terms added to the AdS$_5$ black hole background, this relation holds at first order in the Gauss-Bonnet correction λ_{GB} [?], though exceptions occur at second order in λ_{GB} [?, ?]. Together with the results of [?] and ours, we see that if high-order derivative terms in bulk gravity are not considered, the relation $4\lambda_1 - \lambda_2 = 2\eta\tau_\pi$ has a strong possibility of being universal for both conformal and nonconformal strongly coupled relativistic hydrodynamics.

The dispersion relations are obtained by working in the linearized regime of the fluid [?, ?]:

$$\begin{aligned}
\omega_T(k) &= -i\frac{\eta}{\epsilon+p}k^2 + i\frac{\eta\tau_\pi}{(\epsilon+p)^2}k^4,\\
\omega_L(k) &= \pm c_s k - i\frac{2\eta/3+\zeta/2}{\epsilon+p}k^2 + i\frac{\eta\tau_\pi + \frac{1}{2}\zeta\tau_\Pi}{(\epsilon+p)^2}k^4,
\end{aligned}$$

where "T" and "L" denote transverse (shear) and longitudinal (sound) modes, respectively.

Grozdanov et al.~[?] derived dispersion relations for relativistic fluid at third derivative order. Counting only contributions up to second-order viscous tensors, the dispersion relations for nonconformal fluid up to k^4 are:

$$\begin{aligned}
\omega_T(k) &= -i\frac{\eta}{\epsilon+p}k^2 + i\frac{\eta\tau_\pi}{(\epsilon+p)^2}k^4,\\
\omega_L(k) &= \pm c_s k - i\frac{2\eta/3+\zeta/2}{\epsilon+p}k^2 + i\frac{\eta\tau_\pi + \frac{1}{2}\zeta\tau_\Pi}{(\epsilon+p)^2}k^4.
\end{aligned}$$

Using the first-order (2.15) and second-order (6.10) transport coefficients of our model, one can verify that Eqs.~(6.12) and (6.13) are consistent.

Finally, we discuss causality for the boundary fluid in this paper. According to [?, ?], a relativistic fluid respects causality when the group velocities of both shear and sound modes are less than the speed of light (unity in natural units) in the large-k limit. Using the relevant formulas from [?], we find

$$\frac{\eta\tau_\pi}{\epsilon+p} \simeq 0.54 < 1, \quad \frac{\eta\tau_\pi + \frac{1}{2}\zeta\tau_\Pi}{\epsilon+p} \simeq 0.82 < 1.$$

Thus, the boundary relativistic fluid in our framework is causal.

---

7 Discussions and Outlooks

In this paper, we continue our investigation of second-order transport coefficients for the compactified, near-extremal black D4-brane, building on our previous study [?] via the BDE formalism of fluid/gravity duality [?]. We directly calculate 9 second-order transport coefficients for the nonconformal relativistic fluid living on the boundary. Our work successfully generalizes the BDE formalism to nonconformal backgrounds and provides a new set of directly and analytically calculated second-order transport coefficients for strongly coupled, nonconformal relativistic fluid.

We compare known transport coefficients between the uncharged AdS_5 black hole and the compactified black D4-brane in Table 2 . For the AdS_5 case, we show results from both Green-Kubo and BDE formalism approaches. In the AdS_5 columns, coefficients that exist only in the nonconformal case are marked with "–" . Ref.~[?], based on [?, ?], reformulated the BDE formalism in Weyl-covariant language, allowing the boundary to be a curved spacetime within the same conformal class. This Weyl-covariant version can determine κ for conformal fluids, but it is unknown whether a similar reformulation exists for nonconformal fluids, so we mark this with "?" for the compactified D4-brane case.

Our results cover the entire sector of dynamical second-order transport coefficients. These coefficients satisfy some relations predicted in [?], while the violation of other relations is due to differences in the asymptotic behavior of the bulk metric. Compared to [?], we derive 5 additional second-order coefficients: τ_π^*, τ_Π, and $\xi_{1,2,3}$, which indicate nonconformality. λ_3 and ξ_3 remain zero in this work, similar to the case of λ_3 in [?, ?].

To study transport properties beyond second order, one should start from the complete second-order metric (5.40) and expand it to third order in boundary derivatives, following the same procedure as in this paper. However, this will be very challenging, as according to Grozdanov et al.~[?], a total of 68 new transport coefficients appear at third order for uncharged nonconformal fluid (reducing to 20 in the conformal limit). A fascinating question is whether relations like $4\lambda_1 - \lambda_2 = 2\eta\tau_\pi$ persist at third or higher orders. Recent frameworks for exploring higher-order hydrodynamics [?, ?, ?] may be helpful in this direction. Another approach is the linearized limit [?, ?, ?, ?], though this may not answer questions about nonlinear coefficients like $\lambda_{1,2}$.

Given the literature on second-order strongly coupled hydrodynamics and our achievements in this paper, several aspects merit future exploration. First, one could use the Green-Kubo formula to calculate the thermodynamic second-order coefficients for the background in this paper. Due to its structure, the BDE formalism seems unable to derive thermodynamic transport coefficients in nonconformal backgrounds, but as shown in [?], the Green-Kubo formalism excels at extracting them. We expect to obtain at least κ, κ^*, and ξ_5, not only because they come from 2-point correlation functions and are relatively easier to calculate, but also because these three coefficients form closed constraint equations [?].

Second, calculating the entropy flux would be valuable. While [?, ?, ?] on strongly coupled nonconformal fluids do not mention entropy flux, this subject is accessible in the BDE formalism [?] and represents a good direction for future work.

Third, given the nonconformal version we have developed in [?] and this paper, it is straightforward to calculate second-order coefficients for near-extremal black Dp-branes [?, ?] to test the method of [?].

Finally, it would be interesting to add smeared D0-brane charge to the compactified D4-brane [?] to study nonconformal fluid with a background axial vector charge. The transport coefficients would then depend on both temperature and the theta angle, providing another fascinating subject for investigation.

References

[1] D. H. Rischke, Fluid dynamics for relativistic nuclear collisions, Lect. Notes Phys. 516 (1999) 21, [arXiv:nucl-th/9809044].

[2] L. Rezzolla and O. Zanotti, Relativistic Hydrodynamics, Oxford University Press, 2013.

[3] C. Eckart, The thermodynamics of irreversible processes III. Relativistic theory of the simple fluid, Phys. Rev. 58 (1940) 919.

[4] L. Landau, On the multiparticle production in high-energy collisions, Izv. Akad. Nauk SSSR, Ser. Fiz. 17 (1953) 51.

[5] M. Kranyš, Relativistic hydrodynamics with irreversible thermodynamics without the paradox of infinite velocity of heat conduction, Nuovo Cimento B42 (1966) 51.

[6] P. Vernotte, Compt. Rend. 246 (1958) 3154.

[7] C. Cattaneo, Compt. Rend. 247 (1958) 431.

[8] I. Müller, Zum Paradoxon der Wärmeleitungstheorie, Z. Phys. 198 (1967) 329.

[9] W. Israel, Nonstationary irreversible thermodynamics: A causal relativistic theory, Ann. Phys. 100 (1976) 310.

[10] W. Israel and J. M. Stewart, Thermodynamics of nonstationary and transient effects in a relativistic gas, Phys. Lett. A58 (1976) 213.

[11] W. Israel and J. M. Stewart, Transient relativistic thermodynamics and kinetic theory, Ann. Phys. 118 (1979) 341.

[12] W. A. Hiscock and L. Lindblom, Stability and causality in dissipative relativistic fluids, Ann. Phys. 151 (1983) 466.

[13] W. A. Hiscock and L. Lindblom, Generic instabilities in first order dissipative relativistic fluid theories, Phys. Rev. D 31 (1984) 725.

[14] A. Muronga, Heavy Ion Phys. 15 (2002) 337-347; Proceedings of the 17th Winter Workshop on Nuclear Dynamics, (edited by G. D. Westfall and W. Bauer), (EP Systema, Hungary, 2001).

[15] A. Muronga, Second order dissipative fluid dynamics for ultra-relativistic nuclear collisions, Phys. Rev. Lett. 88 (2002) 062302; Erratum-ibid. 89 (2002) 159901, [arXiv:nucl-th/0104064].

[16] A. Muronga, Ph.D thesis, 2002, unpublished.

[17] A. Muronga, Causal theories of dissipative relativistic fluid dynamics for nuclear collisions, Phys. Rev. C 69 (2004) 034903, [arXiv:nucl-th/0309055].

[18] B. Betz, D. Henkel and D. H. Rischke, Complete second-order dissipative fluid dynamics, J. Phys. G: Nucl. Part. Phys. 36 (2009) 064029.

[19] B. Betz, G. S. Denicol, T. Koide, E. Molnar, H. Niemi and D. H. Rischke, Second order dissipative fluid dynamics from kinetic theory, EPJ Web Conf. 13 (2011) 07005, [arXiv:1012.5772 [nucl-th]].

[20] M. A. York and G. D. Moore, Second order hydrodynamics coefficients from kinetic theory, Phys. Rev. D 79 (2009) 054011, [arXiv:0811.0729 [hep-ph]].

[21] G. D. Moore and K. A. Sohrabi, Kubo formulae for second order hydrodynamic coefficients, Phys. Rev. Lett. 106 (2011) 122302, [arXiv:1007.5333 [hep-ph]].

[22] G. D. Moore and K. A. Sohrabi, Thermodynamical second-order hydrodynamical coefficients, JHEP 1211 (2012) 148, [arXiv:1210.3340 [hep-ph]].

[23] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, Relativistic viscous hydrodynamics, conformal invariance and holography, JHEP 0804 (2008) 100, [arXiv:0712.2451 [hep-th]].

[24] P. Arnold, D. Vaman, Chaolun Wu and Wei Xiao, Second order hydrodynamic coefficients from 3-point stress tensor correlators via AdS/CFT, JHEP 1110 (2011) 033, [arXiv:1105.4645 [hep-th]].

[25] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, Nonlinear fluid dynamics from gravity, JHEP 0802 (2008) 045, [arXiv:0712.2456 [hep-th]].

[26] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita and M. Rangamani, Local Fluid Dynamical Entropy from Gravity, JHEP 0806 (2008) 055, [arXiv:0803.2526 [hep-th]].

[27] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi and S. R. Wadia, Forced fluid dynamics from gravity, JHEP 0902 (2009) 018, [arXiv:0806.0006 [hep-th]].

[28] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, Hydrodynamics from charged black holes, JHEP 1101 (2011) 094, [arXiv:0809.2596 [hep-th]].

[29] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, Fluid dynamics of R-charged black holes, JHEP 0901 (2009) 055, [arXiv:0809.2488 [hep-th]].

[30] M. van Raamsdonk, Black hole dynamics from atmospheric science, JHEP 0805 (2008) 106, [arXiv:0802.3224 [hep-th]].

[31] M. Haack and A. Yarom, Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT, JHEP 0810 (2008) 063, [arXiv:0806.4602 [hep-th]].

[32] M. Haack and A. Yarom, Universality of second order transport from the gauge-string duality, Nucl. Phys. B 813 (2009) 140, [arXiv:0811.1794 [hep-th]].

[33] S. Bhattacharyya, R. Loganayagam, I. Mandal and S. Minwalla, Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions, JHEP 0812 (2008) 116, [arXiv:0809.4272 [hep-th]].

[34] A. Buchel, J. T. Liu and A. O. Starinets, Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory, Nucl. Phys. B 707 (2005) 56, [arXiv:hep-th/0406264].

[35] A. Buchel and M. Paulos, Relaxation time of a CFT plasma at finite coupling, Nucl. Phys. B 805 (2008) 59, [arXiv:0806.0788 [hep-th]].

[36] A. Buchel and M. Paulos, Second order hydrodynamics of a CFT plasma from boost invariant expansion, Nucl. Phys. B 810 (2009) 40, [arXiv:0808.1601 [hep-th]].

[37] S. Grozdanov and A. O. Starinets, On the universal identity in second order hydrodynamics, JHEP 1503 (2015) 007, [arXiv:1412.5685 [hep-th]].

[38] E. Shaverin and A. Yarom, Universality of second order transport in Gauss-Bonnet gravity, JHEP 1304 (2013) 013, [arXiv:1211.1979 [hep-th]].

[39] E. Shaverin, A breakdown of a universal hydrodynamic relation in Gauss-Bonnet gravity, [arXiv:1509.05418 [hep-th]].

[40] S. Grozdanov and A. O. Starinets, Zero-viscosity limit in a holographic Gauss-Bonnet liquid, Theoretical and Mathematical Physics, 182(1) (2015) 61.

[41] F. Bigazzi and A. L. Cotrone, An elementary stringy estimate of transport coefficients of large temperature QCD, JHEP 1008 (2010) 128, [arXiv:1006.4634 [hep-ph]].

[42] S. I. Finazzo, R. Rougemont, H. Marrochio and J. Noronha, Hydrodynamic transport coefficients for the non-conformal QGP from holography, JHEP 1502 (2015) 051, [arXiv:1412.2968 [hep-ph]].

[43] P. Romatschke, Relativistic viscous fluid dynamics and non-equilibrium entropy, Class. Quant. Grav. 27 (2010) 025006, [arXiv:0906.4787 [hep-th]].

[44] H. Grad, On the Kinetic Theory of Rarefied Gases, Comm. Pure Appl. Math. 2 (1949) 331.

[45] R. Kubo, Statistical Mechanical Theory of Irreversible Processes, Journal of the Physical Society of Japan Vol. 12 (1957) 570.

[46] A. Hosoya, M. Sakagami and M. Takao, Nonequilibrium Thermodynamics in Field Theory: Transport Coefficients, Ann. Phys. 154 (1984) 229.

[47] P. Kovtun, Lectures on hydrodynamic fluctuations in relativistic theories, J. Phys. A 45 (2012) 473001, [arXiv:1205.5040 [hep-th]].

[48] D. T. Son and A. O. Starinets, Minkowski-space correlators in AdS/CFT correspondence: recipe and applications, JHEP 0209 (2002) 042, [arXiv:hep-th/0205051].

[49] G. Policastro, D. T. Son and A. O. Starinets, From AdS/CFT correspondence to hydrodynamics, JHEP 0209 (2002) 043, [arXiv:hep-th/0205052].

[50] G. Policastro, D. T. Son and A. O. Starinets, From AdS/CFT correspondence to hydrodynamics. II. Sound waves, JHEP 0212 (2002) 054, [arXiv:hep-th/0210220].

[51] M. P. Heller and R. A. Janik, Viscous hydrodynamics relaxation time from AdS/CFT, Phys. Rev. D 76 (2007) 025027, [arXiv:hep-th/0703243].

[52] E. Barnes, D. Vaman, Chaolun Wu and P. Arnold, Real-time finite-temperature correlators from AdS/CFT, Phys. Rev. D 82 (2010) 025019, [arXiv:1004.1179 [hep-th]].

[53] S. Bhattacharyya, Constraints on the second order transport coefficients of an uncharged fluid, JHEP 1207 (2012) 104, [arXiv:1201.4654 [hep-th]].

[54] I. Kanitscheider and K. Skenderis, Universal hydrodynamics of nonconformal branes, JHEP 0904 (2009) 062, [arXiv:0901.1487 [hep-th]].

[55] R. K. Gupta and A. Mukhopadhyay, On the universal hydrodynamics of strongly coupled CFTs with gravity duals, JHEP 0903 (2009) 067, [arXiv:0810.4851 [hep-th]].

[56] Chao Wu, Yidian Chen and Mei Huang, Fluid/gravity correspondence: A nonconformal realization in compactified D4 branes, Phys. Rev. D 93 (2016) 066005, [arXiv:1508.04038 [hep-th]].

[57] S. Grozdanov and N. Kaplis, Constructing higher-order hydrodynamics: The third order, Phys. Rev. D 93 (2016) 066012, [arXiv:1507.02461 [hep-th]].

[58] P. Romatschke, New developments in relativistic viscous hydrodynamics, Int. J. Mod. Phys. E 19 (2010) 1, [arXiv:0902.3663 [hep-ph]].

[59] R. Loganayagam, Entropy Current in Conformal Hydrodynamics, JHEP 0805 (2008) 087, [arXiv:0801.3701 [hep-th]].

[60] Jian-Hua Gao and Shi Pu, Solving the viscous hydrodynamics order by order, Phys. Rev. C 92 (2015) 044908, [arXiv:1409.1089 [nucl-th]].

[61] F. M. Haehl, R. Loganayagam and M. Rangamani, The eightfold way to hydrodynamic dissipation, Phys. Rev. Lett. 114 (2015) 201601, [arXiv:1412.1090 [hep-th]].

[62] F. M. Haehl, R. Loganayagam and M. Rangamani, Adiabatic hydrodynamics: The eightfold way to hydrodynamic dissipation, JHEP 1505 (2015) 060, [arXiv:1502.00636 [hep-th]].

[63] Yanyan Bu and M. Lublinsky, All Order Linearized Hydrodynamics from Fluid Gravity Correspondence, Phys. Rev. D 90 (2014) 086003, [arXiv:1406.7222 [hep-th]].

[64] Yanyan Bu and M. Lublinsky, Linearized fluid gravity correspondence: from shear viscosity to all order hydrodynamics, JHEP 1411 (2014) 064, [arXiv:1409.3095 [hep-th]].

[65] Yanyan Bu and M. Lublinsky, Linearly resummed hydrodynamics in a weakly curved spacetime, JHEP 1504 (2015) 136, [arXiv:1502.08044 [hep-th]].

[66] Yanyan Bu, M. Lublinsky and A. Sharon, Hydrodynamics dual to Einstein-Gauss-Bonnet gravity, all order gradient resummation, JHEP 1506 (2015) 162, [arXiv:1504.01370 [hep-th]].

[67] J. Mas and J. Tarrío, Hydrodynamics from the Dp-brane, JHEP 0705 (2007) 036, [arXiv:hep-th/0703093].

[68] Chao Wu, Zhiguang Xiao and Da Zhou, Sakai-Sugimoto Model in D0-D4 Background, Phys. Rev. D 88 (2013) 026016, [arXiv:1304.2111 [hep-th]].

Note: Figure translations are in progress. See original paper for figures.