Positive operators of Extended Lorentz cones

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Full Text

Preamble

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Abstract

In this paper, necessary conditions and sufficient conditions are given for a linear operator to be a positive operator of an Extended Lorentz cone. Similarities and differences with the positive operators of Lorentz cones are investigated.

Introduction

The Lorentz cone (second-order cone) is a very important cone in optimization problems. Many models in robust optimization, plant location problems, and investment portfolio management can be formulated as second-order cone programming [?]. In [?] and [?], we generalized the n-dimensional Lorentz cone to mutually dual $p+q$-dimension cones:

$$L(p,q) = \{(x,u) \in \mathbb{R}^p \times \mathbb{R}^q : x \geq \|u\| e\}$$

$$M(p,q) = \{(x,u) \in \mathbb{R}^p \times \mathbb{R}^q : \langle x, e\rangle \geq \|u\|, x \geq 0\}$$

where $e = (1, \ldots, 1) \in \mathbb{R}^p$. The extended Lorentz cones $L(p,q)$ and $M(p,q)$ become Lorentz cones exactly in the special case $p = 1$. This is the only case when $L(p,q)$ is self-dual.

The set $\Gamma(C)$ of positive operators of a cone C is defined by $\Gamma(C) = \{A \in \mathbb{R}^{(p+q)\times(p+q)} : AC \subseteq C\}$ [?]. The set of positive operators is a cone in $\mathbb{R}^{n\times n}$ [?]. It can be easily checked that A is a positive operator of C if and only if A^T is a positive operator of C^*. The authors of [?] introduced the characteristic matrix of the Lorentz cones as $J_n = \mathrm{diag}(-1, \ldots, -1, 1)$, and showed that the Lorentz cone can be represented as:

$$L(p,1) = \{x \in \mathbb{R}^n : x^T J_n x \geq 0 \text{ and } x_n \geq 0\}.$$

Moreover, they proved the following theorem which characterizes a positive operator by a positive semidefiniteness condition [?]:

Theorem 1. Let $A \in \mathbb{R}^{n\times n}$. If $A \in \Gamma(L(p,1)) \cup \Gamma(-L(p,1))$, then there exists a $\mu \geq 0$ such that $A^T J_n A \succeq \mu J_n$. Conversely, if $\mathrm{rank}(A) \neq 1$ and there is a $\mu \geq 0$ such that the above holds, then $A \in \Gamma(L(p,1)) \cup \Gamma(-L(p,1))$.

Since $L(p,q)$ and $M(p,q)$ are extensions of the second-order cone, the problem of finding the positive operators of $L(p,q)$ and $M(p,q)$ arises naturally. The aim of this short note is to find both the necessary conditions and sufficient conditions for a linear operator to be a positive operator of $L(p,q)$ or $M(p,q)$ and state the similarities and differences between the case $p = 1$ and $p > 1$. In [?], Sznajder determines all automorphism operators of $L(p,q)$. In particular, these operators are also positive operators of $L(p,q)$. This shows that the problem of finding all positive operators of $L(p,q)$ (or $M(p,q)$) is a much more difficult problem than the one solved by Sznajder. Although this problem is still open, the present note presents some partial results by finding necessary conditions and sufficient conditions for a linear operator to be a positive operator of $L(p,q)$ (or $M(p,q)$).

The structure of the paper is as follows. First we introduce some notation. Then, we will prove a lemma about the characterization of an extended Lorentz cone. Finally, based on this lemma, we present necessary conditions and sufficient conditions for a linear operator to be a positive operator of an extended Lorentz cone.

2 Notations

Denote by $\mathbb{N}$ the set of positive integers. Let $k, m, p, q \in \mathbb{N}$ and $\mathbb{R}^m$ be the m-dimensional real Euclidean vector space. Define the direct product space $\mathbb{R}^p \times \mathbb{R}^q$ as the set of all pairs of vectors (x, u), where $x \in \mathbb{R}^p$ and $u \in \mathbb{R}^q$. Identify the vectors of $\mathbb{R}^m$ by column vectors and consider the canonical inner product defined on $\mathbb{R}^m$ by $\mathbb{R}^m \times \mathbb{R}^m \ni (x, y) \mapsto \langle x, y\rangle := x^\top y \in \mathbb{R}$ with the induced norm $\mathbb{R}^m \ni x \mapsto \|x\| = \sqrt{\langle x, x\rangle} \in \mathbb{R}$. The inner product in $\mathbb{R}^p \times \mathbb{R}^q \equiv \mathbb{R}^{p+q}$ is given by $((x, u), (y, v)) \mapsto \langle x, y\rangle + \langle u, v\rangle$.

A closed and convex set K is said to be a cone if $K \cap (-K) = \{0\}$, and $\lambda v \in K$, $\forall \lambda \geq 0$ and $\forall v \in K$. A cone K is said to be a proper cone if the interior of K is nonempty. Note that both $L(p, q)$ and $M(p, q)$ are proper cones. For any cone $K \subseteq \mathbb{R}^m$, its dual cone is defined by $D := K^* := \{y \in \mathbb{R}^m : \langle x, y\rangle \geq 0, \forall x \in K\}$. We also have that $D^* = K$, so D and K are mutually dual. The complementarity set of K is defined by $C(K) = \{(x, s) : x \in K, s \in K^*, \langle x, s\rangle = 0\}$ [**?**].

A matrix $A \in \mathbb{R}^{m\times m}$ is said to be Lyapunov-like on K if $\langle Ax, s\rangle = 0$ for all $(x, s) \in C(K)$. Such matrices (transformations) are also characterized by the condition $e^{tA} \in \mathrm{Aut}(K)$ for all $t \in \mathbb{R}$, where $\mathrm{Aut}(K)$ denotes the automorphism group of the cone K. Note that any automorphism is a positive operator.

3 Main Results

First we need to introduce two lemmas.

Lemma 1. Let $M = M(p, q)$ be the dual cone of the extended Lorentz cone $L(p, q)$. Then,

$$M = \{z = (x, u) \in \mathbb{R}^p \times \mathbb{R}^q : z^T J z \geq 0, x \in \mathbb{R}^p_+\}$$

where

$$J = \begin{pmatrix} E & 0 \\ 0 & -I \end{pmatrix}$$

and E is a $p \times p$ matrix with all entries equal to 1.

Proof. We have $z = (x, u) \in M$ if and only if $z^T J z \geq 0$ and $x \geq 0$, or equivalently $0 \leq z^T J z = x^T E x - u^T u = \langle x, e\rangle^2 - \|u\|^2$ and $x \geq 0$. Thus, $z = (x, u) \in M$ if and only if $\langle x, e\rangle \geq \|u\|$ and $x \geq 0$. □

The next theorem states necessary conditions for a linear operator to be a positive operator of $M(p, q)$.

Theorem 2 (Necessary Conditions for Positive Operators of $M(p, q)$). Let $p > 1$ and $q > 0$ be integers. Let

$$A = \begin{pmatrix} W & R \\ S & T \end{pmatrix} \in \mathbb{R}^{(p+q)\times(p+q)},$$

where $W \in \mathbb{R}^{p\times p}$, $R \in \mathbb{R}^{p\times q}$, $S \in \mathbb{R}^{q\times p}$ and $T \in \mathbb{R}^{q\times q}$. If A is a positive operator of $M(p,q)$, then the following holds:

(i) The first p rows of A (more precisely their transpose) are in $L(p,q)$.

(ii) The first p columns of A are in $M(p,q)$.

(iii) By adding any i-th column, $i = 1,\dots,p$, to the linear combination of the columns $p+1,\dots,p+q$ with coefficients $u_1,\dots,u_q$ such that the Euclidean norm of $u = (u_1,\dots,u_q)^T$ is one, we obtain an element in $M(p,q)$.

(iv) The sum of any i-th column, $i = 1,\dots,p$, with any $(p+j)$-th column, $j = 1,\dots,q$, is in $M(p,q)$.

(v) If A is M-Lyapunov-like, then $e^{tA} \in \mathrm{Aut}(M(p,q))$ and hence it is in particular a positive operator of $M(p,q)$, for any $t \in \mathbb{R}$.

Proof. (i) Since A is a positive operator of $M(p,q)$, the first p entries of Az are nonnegative for any $z \in M(p,q)$. Hence, the inner product of z and any row vector of the first p rows is nonnegative. Therefore these row vectors must be in the dual cone of $M(p,q)$, that is, $L(p,q)$.

(ii) Based on the above argument, A^T is a positive operator of $L(p,q)$; hence (ii) follows similarly to (i).

(iii) Let $\beta_i = \alpha_i + \sum_{j=1}^q u_j\alpha_{p+j}$, where α_t is the t-th column of A. Then for any $z \in L(p,q)$,

$$\langle z,\beta_i\rangle = \langle z,\alpha_i\rangle + \sum_{j=1}^q u_j\langle z,\alpha_{p+j}\rangle.$$

By the Cauchy-Schwarz inequality, we have:

$$\left|\sum_{j=1}^q u_j\langle z,\alpha_{p+j}\rangle\right| \le \|u\|\sqrt{\sum_{j=1}^q \langle z,\alpha_{p+j}\rangle^2} = \sqrt{\sum_{j=1}^q \langle z,\alpha_{p+j}\rangle^2}.$$

Therefore,

$$\langle z,\beta_i\rangle \ge \langle z,\alpha_i\rangle - \sqrt{\sum_{j=1}^q \langle z,\alpha_{p+j}\rangle^2}.$$

Since A is a positive operator of $M(p,q)$, A^T is a positive operator of $L(p,q)$ and therefore $A^Tz \in L(p,q)$. Since $\langle z,\alpha_k\rangle$ is the k-th entry of A^Tz, by the definition of $L(p,q)$, we have that the right-hand side of the above inequality is nonnegative. Thus, $\beta_i \in M(p,q)$.

(iv) Obviously, this is a special case of the above assertion.

(v) See [?]. □

Theorem 3 (Sufficient Condition for Positive Operators). If there exists a $\lambda \geq 0$ such that $A^T JA - \lambda J$ is positive semidefinite and the first p rows of A are in $L(p,q)$, then A is a positive operator of $M(p,q)$.

Proof. Since the first p rows of A are in $L(p,q)$, the first p entries of Az are nonnegative for any $z \in M(p,q)$. Since $A^T JA - \lambda J$ is positive semidefinite, we have

$$0 \leq z^T(A^T JA - \lambda J)z = (Az)^T J(Az) - \lambda z^T Jz.$$

By Lemma 1, $\lambda z^T Jz \geq 0$. So $(Az)^T J(Az) \geq 0$, hence by Lemma 1, $Az \in M(p,q)$. Thus, we have that A is a positive operator of $M(p,q)$. □

Similar necessary conditions and sufficient conditions can be given for the positive operators of $L(p,q)$.

4 Conclusion

In this paper, we established necessary conditions and sufficient conditions for positive operators of Extended Lorentz cones. Since the first p entries (rather than only the first entry of vectors in $L(1,q)$) must all be positive, some extra conditions (such as the first p rows being in $L(p,q)$) are needed to ensure that A is a positive operator when $A^T JA - \lambda J$ is positive semidefinite.

In the future, we wish to find a necessary and sufficient condition for a linear operator to be a positive operator of an extended Lorentz cone.

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Note: Figure translations are in progress. See original paper for figures.