Preliminary Exploration of a Novel Difference Model for the Mathematical Principles of the Pirate Gold Division Problem

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Abstract

Since the introduction of the intriguing Pirate Game (PG) problem in game theory, its theoretical analysis, Only abstract: Since the introduction of the intriguing Pirate Game (PG) problem in game theory, its theoretical analysis has been confined to backward induction and sequential recursion methods. This paper first establishes a first-order difference model by drawing upon these two methods; subsequently, considering that each pirate is not perfectly rational and that higher-ranked pirates must rely on the decisions of their immediate subordinates to make optimal choices, a second-order delay difference model is constructed to conduct an in-depth mathematical analysis of PG: when = 0, the model exhibits significant deviation from practical scenarios; when the delay parameter =1, the model' s solution coincides with that of the first-order difference model, suggesting that in real-world situations there exist social groups that directly consult their first deputies when making decisions. Consequently, at the level of modern analytical methodologies, this paper presents a novel and reasonable mathematical interpretation of PG.

Full Text

Preamble

A Preliminary Study on a New Difference Model for the Mathematical Principles of the Pirate Game

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Abstract

Since the intriguing Pirate Game (PG) was introduced in game theory, theoretical analysis has been limited to backward induction and series recursion methods. This paper first establishes a first-order difference model based on these two methods. Then, considering that each pirate is not absolutely rational and that high-ranking pirates must rely on the decisions of pirates one rank lower to make optimal decisions, we construct a second-order delay difference model to conduct an in-depth analysis of the PG from a mathematical principle perspective: when the time delay is 0, the deviation from reality is significant; when the time delay is 1, the solution of the model coincides with that of the first-order difference model, indicating that in real life there exist social groups where decision-makers directly consult their first deputies. Thus, this paper provides a new and reasonable mathematical explanation for the PG at the level of modern analytical methods.

Keywords: Pirate game; first-order difference model; second-order difference model; delay
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1. Introduction

The Pirate Game (PG) is a long-standing puzzle [1-4] that describes how five perfectly rational pirates propose schemes to divide 100 gold coins, with each proposer seeking maximum benefit. In a strictly hierarchical pirate society, proposals must follow the class system. If a proposal receives support from half or more of the pirates, it is implemented; otherwise, the proposer is thrown into the sea and the highest-ranking remaining pirate makes a proposal, continuing until only one pirate remains [1]. In 1999, Ian Stewart solved this puzzle [1, 2] by first numbering the five pirates such that higher rank corresponds to a larger number, i.e., Pirate 5 has the highest rank. Using backward induction, the optimal allocation proposed by the highest-ranking pirate (Pirate 5) is: Pirate 1 receives 1 coin, Pirate 2 receives 0 coins, Pirate 3 receives 1 coin, Pirate 4 receives 0 coins, and Pirate 5 keeps 98 coins for himself. Stewart later extended this analysis to 500 pirates.

To date, including methods presented on Wikipedia, only backward induction and series recursion approaches [3, 5] have been available, with no more reasonable and in-depth mathematical models or complete theoretical frameworks [2]. This paper employs modern mathematical analysis methods to construct more reasonable mathematical models, namely a first-order difference model, thereby advancing the development of PG theory.

2. Series Recursion Method

Assume that n pirates divide 100 gold coins, where $n \leq 200$. In this case, the highest-ranking pirate can propose a solution that secures both his life and gold. However, when the number of pirates increases further, the number of gold coins becomes insufficient for the highest-ranking pirate to buy support. To survive, he may receive no gold coins at all.

When $n = 201$, i.e., when there are 201 pirates, Pirate 201 needs support from 101 or more pirates to pass his proposal. Therefore, he cannot keep any gold coins but must distribute all 100 gold coins to other pirates with odd numbers, with each receiving exactly one coin.

Let $\{x(n)\}$ denote the sequence of pirate numbers who survive but receive no gold when it is their turn to make decisions [8]. Let the first pirate who survives without gold be $x(0) = 201$. Based on the above analysis, for a pirate numbered $x(n)$, to pass his proposal he needs support from half or more pirates, i.e., $x(n)/2$ pirates. Since 100 gold coins can be used to bribe 100 pirates, the actual consideration should be securing votes from all pirates numbered between $x(n-1)$ and $x(n)$. However, these votes are exactly $x(n-1)$. Thus, the recurrence relation [5] is:

$$x(n) - x(n-1) = 100 - 2x(n-1)$$

Solving this yields:

$$x(n) = 200 + 2n, \quad n = 0, 1, 2, ...$$

In plain terms: when $n = 202$, i.e., when there are 202 pirates, Pirate 202 needs support from 101 or more pirates. He can only distribute all gold coins to pirates who received no gold when $n = 201$, excluding Pirate 201. When $n = 203, 204, ...$, the pattern continues similarly.

3. First-Order Difference Model

Based on the series recursion method and drawing from difference equation theory [6-8], this paper develops the following framework.

Definition 1 [6]: Let n be a non-negative integer. The first-order backward difference is defined as:

$$\nabla x(n) = x(n) - x(n-1)$$

The first-order difference model constructed in this paper is:

$$\begin{cases} x(0) = 201 \\ \nabla x(n) - x(n) = -\dfrac{100}{2} \end{cases}$$

The corresponding first-order homogeneous difference equation [8] is:

$$\nabla x(n) - x(n) = 0$$

Its particular solution satisfies:

$$x(n) - x(n-1) = \frac{100}{2} - x(n-1)$$

For the homogeneous difference equation, let the characteristic value be λ. The characteristic equation [9] is:

$$\lambda - 1 = 0$$

The general solution of the homogeneous difference equation is $x_h(n) = c \cdot 1^n$, where c is an arbitrary constant. Therefore, the general solution of equation (2) is:

$$x(n) = \frac{200}{2} + c \cdot 1^n$$

Thus, the solution of the first-order difference system (1) is:

$$x(n) = 200 + 2n$$

When $x(0) = 202$, the solution of the first-order difference system is $x(n) = 200 + 2n + 1$. This does not affect the solution of the difference system, nor does it impact subsequent research on the second-order delay difference model.

4. Second-Order Delay Difference Model

The Pirate Game model represents a perfect theoretical framework, but real-world social problems are complex and variable. Pirates who are completely rational and emotionless do not exist, and some social group leaders make decisions by directly consulting their first deputies. Therefore, based on the first-order difference model, we transform system (1) as follows:

$$\begin{cases} y(0) = 201 \\ y(1) = 202 \\ \nabla^2 y(n) = \dfrac{100}{2} - y(n-\tau) \end{cases}$$

This yields the second-order delay difference system [10-12], where τ is the delay parameter and $\tau \geq 0$. When $\tau < 2$, additional initial values are required, which lacks practical significance. When $\tau \geq 2$, the solution process proceeds as follows.

4.1. Case $\tau = 0$

When $\tau = 0$, the system becomes:

$$\begin{cases} y(0) = 201 \\ y(1) = 202 \\ \nabla^2 y(n) = \frac{100}{2} - y(n) \end{cases}$$

This is a second-order linear difference system with constant coefficients. From Definition 1, we have:

$$\nabla^2 y(n) = \nabla(\nabla y(n)) = \nabla(y(n) - y(n-1)) = [y(n) - y(n-1)] - [y(n-1) - y(n-2)]$$

Simplifying gives:

$$y(n) - 2y(n-1) + y(n-2) = \frac{100}{2} - y(n)$$

Rearranging:

$$2y(n) - 2y(n-1) + y(n-2) = \frac{100}{2}$$

Equation (4) is a linear nonhomogeneous difference equation with constant coefficients. Let its particular solution be $y_p(n) = b$, then:

$$2b - 2b + b = \frac{100}{2}$$

Thus $b = 200$. The corresponding homogeneous difference equation is:

$$2y(n) - 2y(n-1) + y(n-2) = 0$$

Its characteristic equation is:

$$2\lambda^2 - 2\lambda + 1 = 0$$

The two real roots are $\lambda_1 = 2 + \sqrt{2}$ and $\lambda_2 = 2 - \sqrt{2}$. The general solution of equation (4) is:

$$y(n) = c_1(2 + \sqrt{2})^n + c_2(2 - \sqrt{2})^n + 200$$

Using the initial values, we obtain the system:

$$\begin{cases} c_1 + c_2 = 1 \\ c_1(2+\sqrt{2}) + c_2(2-\sqrt{2}) = 2 \end{cases}$$

Solving yields $c_1 = c_2 = \frac{1}{2}$. Substituting these constants into the general solution gives the solution of the second-order difference system (3):

$$y(n) = 200 + \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{2}$$

Define the difference $z(n) = y(n) - x(n)$:

$$z(n) = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{2} - 2n$$

[Figure 1: see original paper] Difference between solutions of systems (1) and (3)

Figure 1 shows that as n increases, the value of $z(n)$ grows larger. Therefore, the second-order difference model without delay is unsuitable for describing this practical scenario, meaning the second-order difference model (without delay) cannot solve the Pirate Game. Consequently, this paper considers the second-order delay difference model.

4.2. Case $\tau = 1$

When $\tau = 1$, the second-order delay difference model becomes:

$$\begin{cases} y(0) = 201 \\ y(1) = 202 \\ \nabla^2 y(n) = \frac{100}{2} - y(n-1) \end{cases}$$

Equation (6) is a linear nonhomogeneous difference equation with constant coefficients. Let its particular solution be $y_p(n) = b$, then:

$$b = \frac{100}{2} - b$$

Thus $b = 200$. The corresponding homogeneous difference equation is:

$$5y(n) - y(n-1) + y(n-2) = 0$$

Its characteristic equation is:

$$5\lambda^2 - \lambda + 1 = 0$$

The two real roots are $\lambda_1 = \lambda_2 = \frac{1}{2}$. The general solution of equation (6) is:

$$y(n) = c_1 \left(\frac{1}{2}\right)^n + c_2 n \left(\frac{1}{2}\right)^n + 200$$

Using the initial values yields the system:

$$\begin{cases} c_1 = 1 \\ c_1 \cdot \frac{1}{2} + c_2 \cdot \frac{1}{2} = 2 \end{cases}$$

Solving gives $c_1 = 1$ and $c_2 = 1$. Substituting these constants into the general solution yields the solution of the second-order delay difference system (5):

$$y(n) = 200 + 2n$$

This result is completely consistent with the analytical solution of the first-order difference model. Therefore, the second-order delay difference model (with delay parameter $\tau = 1$) can describe the Pirate Game.

5. Conclusion

In complex and variable real-life situations, individuals are not perfectly rational, and higher-ranking individuals often make optimal decisions based on lower-ranking individuals' choices to secure their positions and satisfy collective demands.

This paper, grounded in difference equation theory, attempts to analyze and theoretically extend the PG in game theory. First, modern mathematical analysis methods are employed to construct a more reasonable mathematical model —the first-order difference model. Second, recognizing that not everyone is absolutely rational in real society and that other social factors influence decisions, this paper further improves upon the first-order difference system by constructing a second-order delay difference system and providing a deep mathematical explanation from theoretical principles. When the delay parameter is 1, the solution of this system completely coincides with that of the first-order difference system, demonstrating that high-ranking individuals depend on the strategies of those one rank lower to make optimal decisions. In real life, some social group leaders indeed exhibit the phenomenon of directly consulting their first deputies when making decisions.

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Note: Figure translations are in progress. See original paper for figures.